Determining a function that describes the relationship between a height of a light source and an area of a shadow cast by a prolate spheroid.

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1. Introduction

The problem explored in this paper will be fully theoretical however, it is based on a real situation. Given a light source above an object placed on some surface in otherwise completely dark room the given object will cast a shadow. A particular case of this phenomenon can be observed when a lamp is placed above a rugby ball causing an elliptical shadow to be cast. This leads to a question how could the area of this elliptical shadow be modeled in the relation to the height of the lamp. The first question that needs to be answered when considering such a question is what theoretical geometric figure would best represent a rugby ball. The object that chosen to represent the rugby ball was a prolate spheroid as it best fitted its shape. Spheroids also known as ellipsoids of revolution are obtained by rotating a ellipse about one of its axes (Torge, Müller, 2012). In this context it is important to define what is a ellipse and what is it's equation that describes it in the Cartesian plane. Ellipses are part of a set of non-degenerate conics. "The non-degenerate conics can be defined as the set of points P in the plane that satisfy the following condition: The distance of P from a fixed point (the focus) is a constant multiple e (the eccentricity) of the distance of P from a fixed line (the directrix)."(Brennan, Esplen, Gray, 2012).

This definition is quite complex and needs further explanation. Firstly conic sections such as a ellipse have a center. This means that there is a point C such that a rotation about this point by the angle π is a symmetry of the conic (Brennan, Esplen, Gray, 2012). Secondly, we define a constant value e as the eccentricity. The eccentricity is the deviation of the shape of a conic section from a circle. The eccentricity is of a circle is 0. An ellipse that is not a circle is defined to have a value of eccentricity between 0 and 1. Thirdly we define the point C as the midpoint of the line segment joining two fixed points defined as $F1$ and $F2$. These points are the foci of the ellipse. Now we define a line segment passing through these two points with a constant distance defined as $2a$ which is greater that the distance between $F1$ and $F2$ and it's midpoint being point *C* and also being colinear with *F1*and *F2*. Basing on this information we get that the coordinates of *F1* are *(ae,0)* and analogously the coordinates of *F2* are *(-ae,0)*. Now we define a vertical line (directrix) as $x = \frac{a}{e}$ and a point *M* on that line. Now from the definition of the ellipse we know that the distance between *P* and *M* is a constant multiple of the eccentricity. Hence:

$$FP = e \cdot PM$$

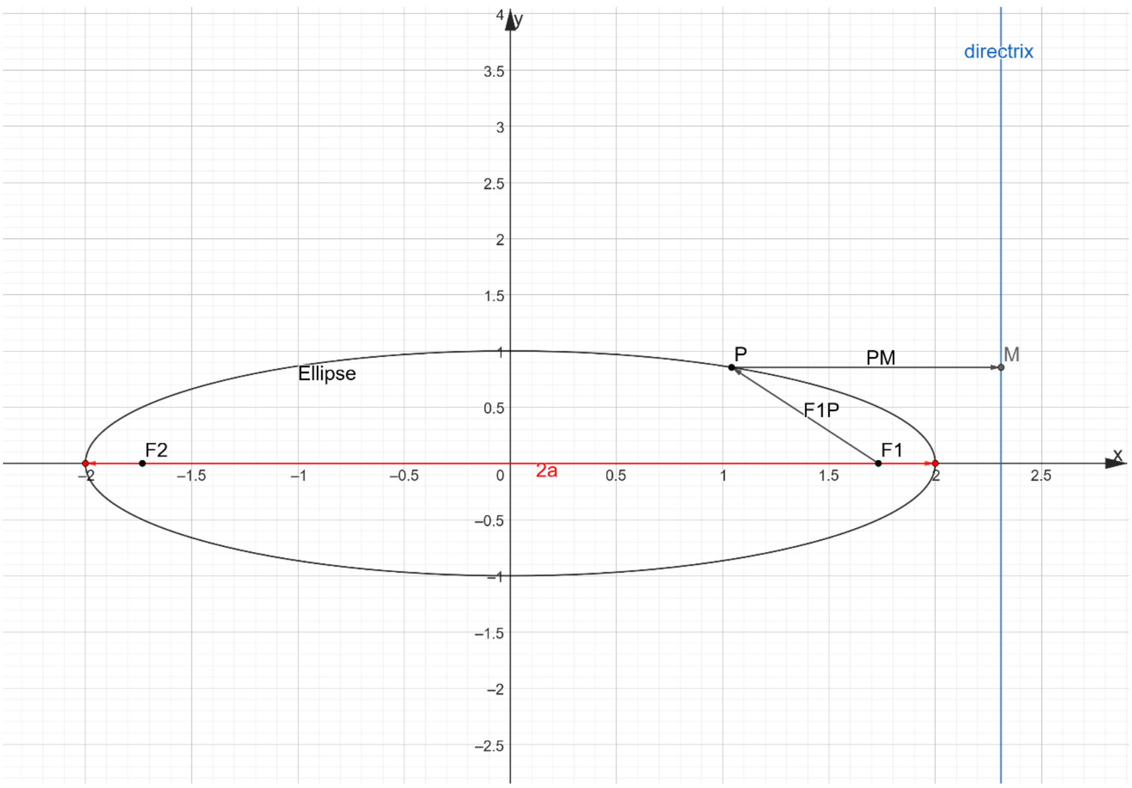

Figure 1.: 2 dimensional graphic of an ellipse with the equation: $\frac{x^2}{2^2} + \frac{y^2}{1^2} = 1$ *, considered in the exploration (all figures created using GeoGebra analytical software)*

Knowing all this we derive the standard equation of an ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$$

If we substitute a coefficient *b* for $a^2(1-e^2)$ we will obtain:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

In the case of this exploration the values of *a* and *b* are:

$$\begin{cases} a = 2 \\ b = 1 \end{cases}$$

Due to the theoretical nature of this exploration these values are chosen arbitrarily.

Hence the equation of the ellipse considered in the exploration is given by:

$$\frac{x^2}{2^2} + \frac{y^2}{1^2} = 1$$

The eccentricity of any ellipse is given by:

$$e = \sqrt{1 - \frac{b^2}{a^2}}$$

The equation of eccentricity can be simply derived from the standard equation of an ellipse.

By substituting the chosen values of *a* and *b* we obtain the eccentricity of the ellipse considered in the exploration:

$$e = \sqrt{1 - \frac{1^2}{2^2}}$$

$$e = \sqrt{\frac{3}{4}}$$

We can now approximate this by using the GeoGebra online scientific calculator (https://www.geogebra.org/scientific?lang=pl), all other approximations are also computed using this software:

$$e \approx 0{,}866 \; (3 \; s.f.)$$

This value fits the previously stated value of eccentricity defining an ellipse. The value of the eccentricity is rounded to 3 significant figures as per convention.

Now we must answer the question how to accurately represent the prolate spheroid in 3 dimensional space.

To answer this question firstly we need to answer the question what the x-axis, y-axis and z-*axis* represents. We will define the x-axis as the length, y-axis as the height and z-axis width. Length, height and width are distances hence are real non-negative numbers. Secondly given a standard equation of an ellipse in terms of the semi-major axis a and semi-minor axis b. We can extend this concept into a 3 dimensional space by rotating the ellipse about its semi-major axis a. This procedure leads us to obtain a prolate spheroid. In figure 2 we can see the graphic of the prolate spheroid with set values which will be investigated in the exploration. Equation of the prolate spheroid considered in the exploration:

$$\frac{x^2}{2^2}+\frac{(y-1)^2}{1^2}+\frac{z^2}{1^2}=1$$

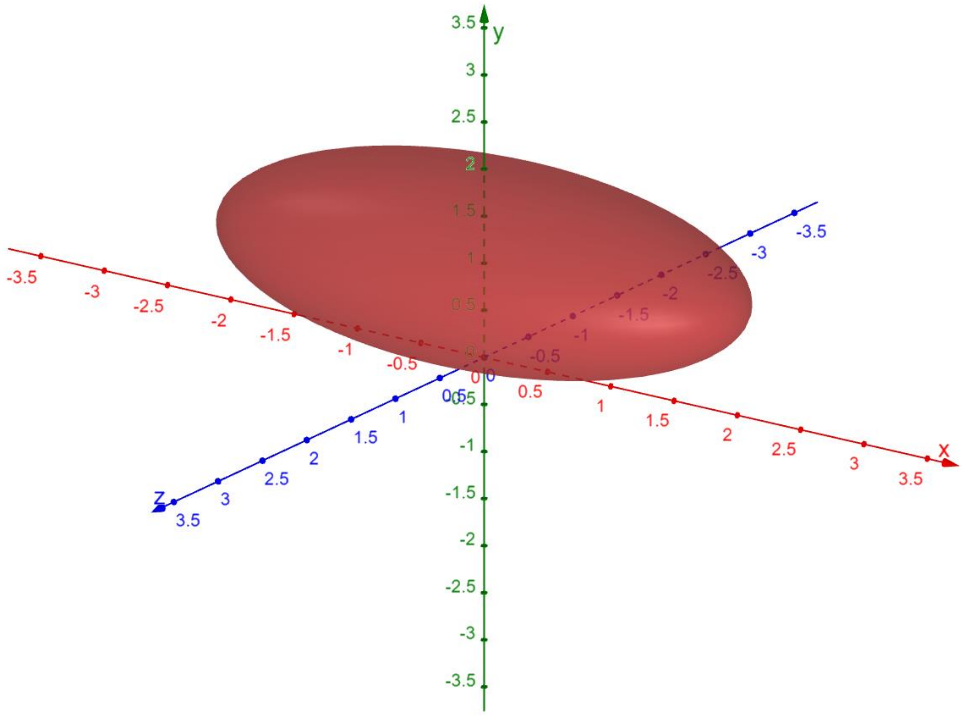

Figure 2.:3 dimensional graphic of a prolate spheroid with the equation: $\frac{x^2}{2^2}+\frac{(y-1)^2}{1^2}+\frac{z^2}{1^2}=1$

This "MM's" like shape is the prolate spheroid. Its semi-major axis a is parallel to the x-axis while the two semi-minor axes (b_1=1; b_2=1) are parallel to the z-axis and y-axis respectively. The spheroid defined by the previous equation and shown in figure 2 is also translated by a vector v_1.

$$\overrightarrow{\boldsymbol{v_1}}=\begin{pmatrix}0\\1\\0\end{pmatrix}$$

This translation is done to fit the real world interpretation of this problem because if the ellipse was only rotated by its semi-major axis half of the spheroid would be under XZ-plane that is considered the "ground" in this hypothetical problem hence it is translated by such a vector v_1. The entire spheroid has positive y-coordinates and is tangent to the XZ-plane in the point with the smallest y-coordinate. These restrictions on the problem provide a clear interpretation of where the spheroid is and what are its dimensions.

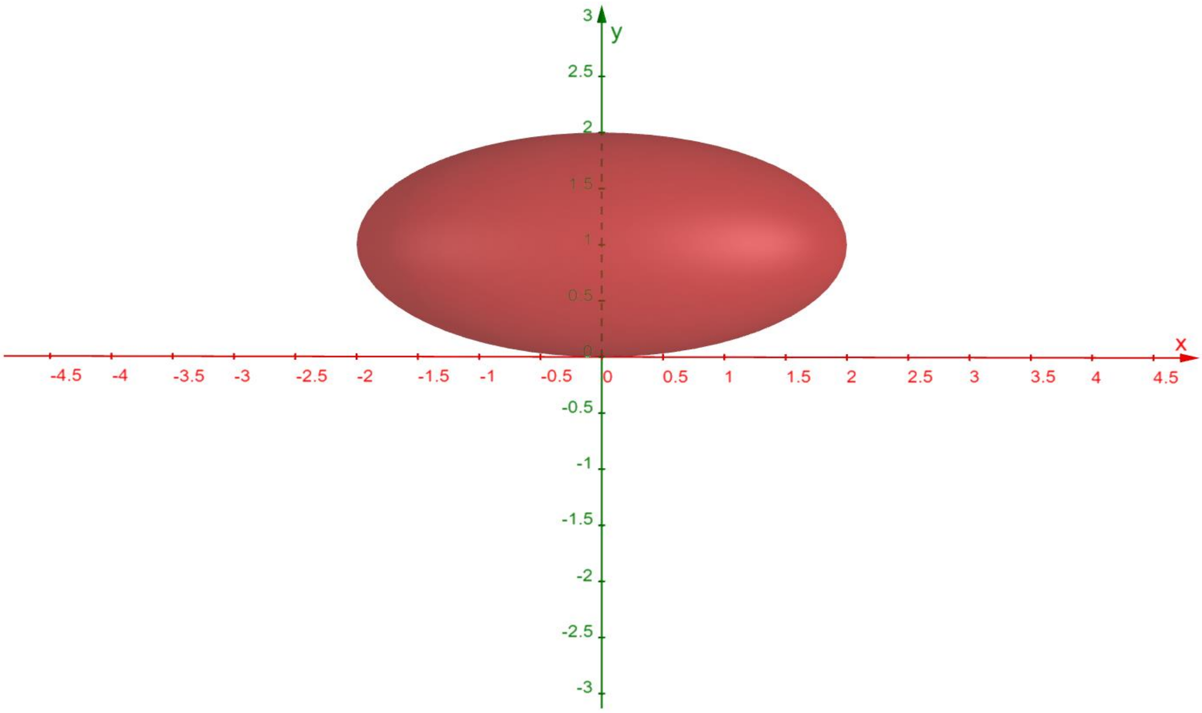

Figure 3.: 2 dimensional cross-section along the x-axis of the prolate spheroid

In figure 3 we see the prolate spheroid "cut" along the x-axis giving us a clear view of the original ellipse used. We see that the dimensions of the spheroid being 4 units in length and 2 units in height.

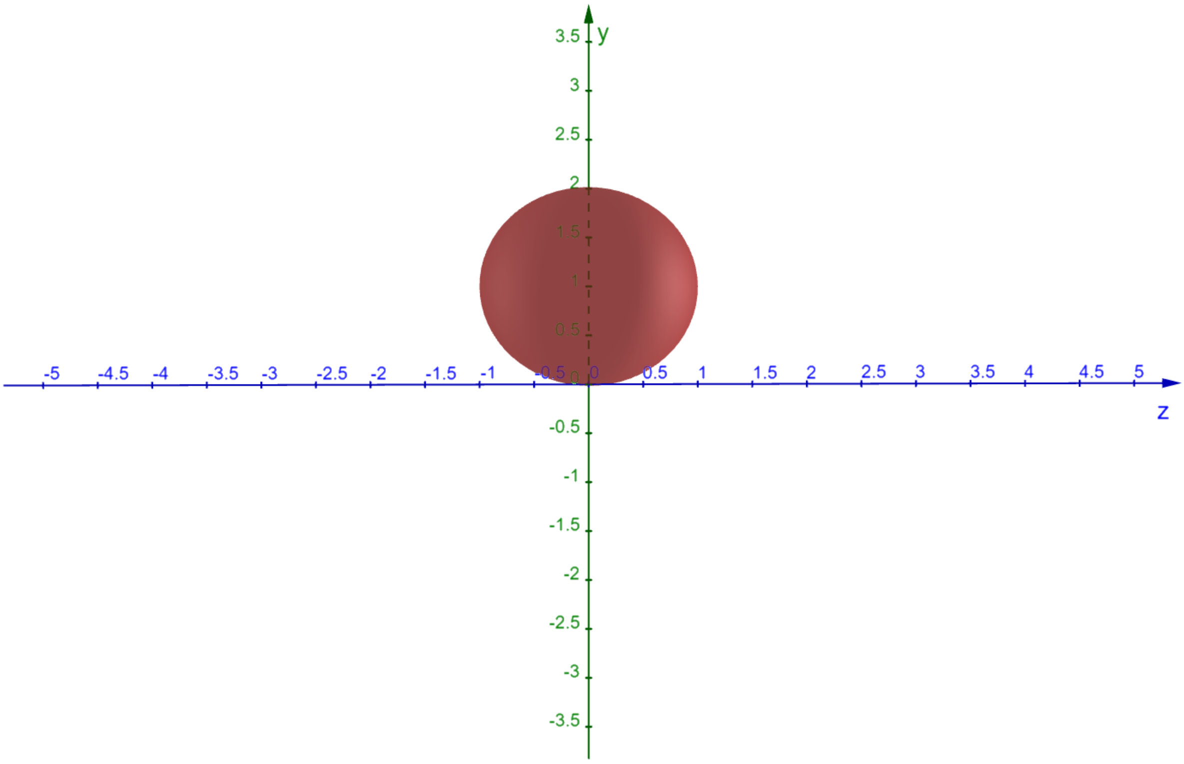

Figure 4.: 2 dimensional cross-section along the z-axis of the prolate spheroid

In figure 4 we see the spheroid being “cut” again this time along *z*-axis giving a clear view of its dimensions. Being 2 units width and 2 units in height which matches up with the previous cross-section.

The overall dimensions of the sphere are 4 units in length (*x*-axis), 2 units in width(*z*-axis), 2 units in height (*y*-axis). In the cross-section along the *x*-axis the spheroid is a ellipse and in a cross-section along the *z*-axis it is a circle.

The equation of the ellipse in the *x*-axis cross-section:

$$\frac{x^2}{2^2} + \frac{(y-1)^2}{1^2} = 1$$

The equation of the circle in the *z*-axis cross-section:

$$x^2 + (y-1)^2 = 1$$

Now we must answer the question how to accurately model the position of the light above the spheroid and how to approach the modelling of the “light rays” casting the elliptical shadow on the *XZ*-plane.

The equations describing the cross-sections will be used to calculate the lengths of the semi-major and semi-minor axes of the shadow to obtain its area.

Now we must consider the light source. It is important to establish the position of the light source and how its height will be manipulated.

The light source in the 3 dimensional space is assumed to be a point L. This point lies on the y-axis and has coordinates $L(0,\ m{+}2,\ 0)$. The parameter m is introduced as a variable describing the relative height of the point to the spheroid.

To obtain the “shadow” of the spheroid we draw tangent lines from the point L to the spheroid. We also have to assume that the parameter m has to be positive as distance cannot be negative. We define that:

$$m \in (0, +\infty)$$

In figure 5 we can see the visualization of the “light rays” being tangent lines from point L to the prolate spheroid.

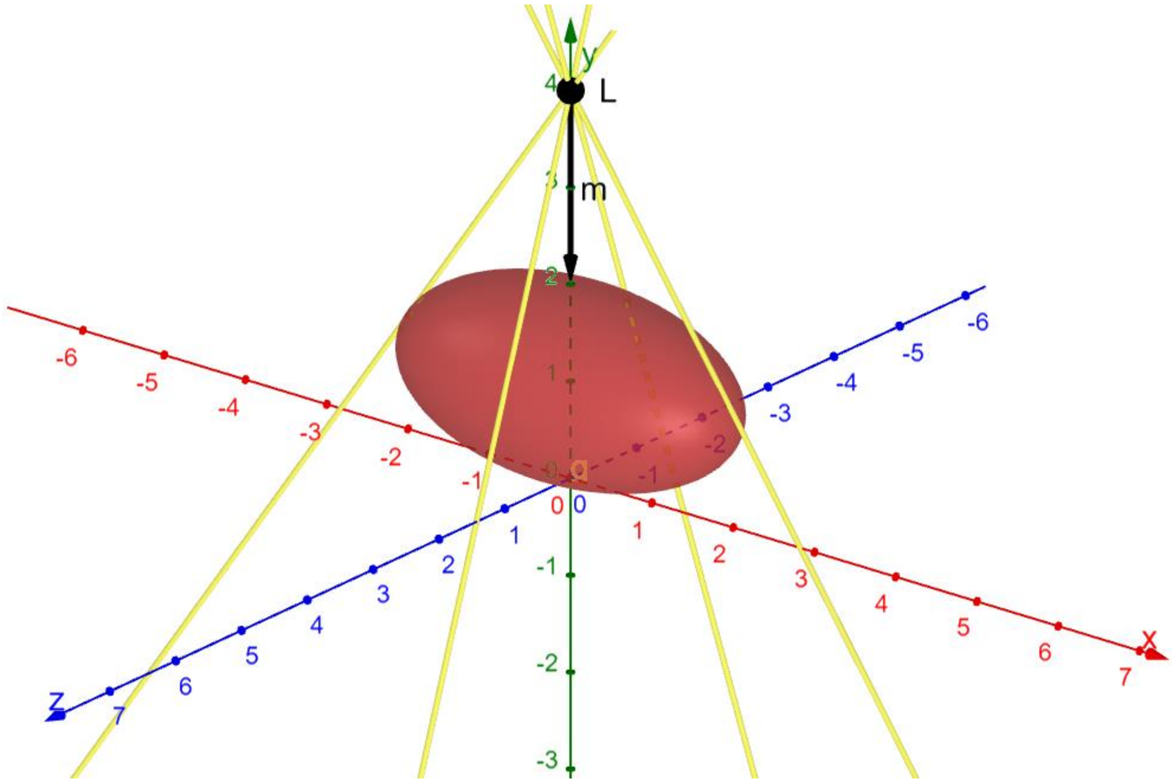

Figure 5.: 3 dimensional visualization of 4 "light rays" being tangent lines and the distance parameter m

The function describing the shadow of the spheroid will be dependent on the value of the parameter m. As m increases the area of the shadow will decrease. As m decreases the area of shadow will increase. In figure 6 we can see the ellipse created by the light rays and with its semi-minor axis marked as D and the semi-major F.

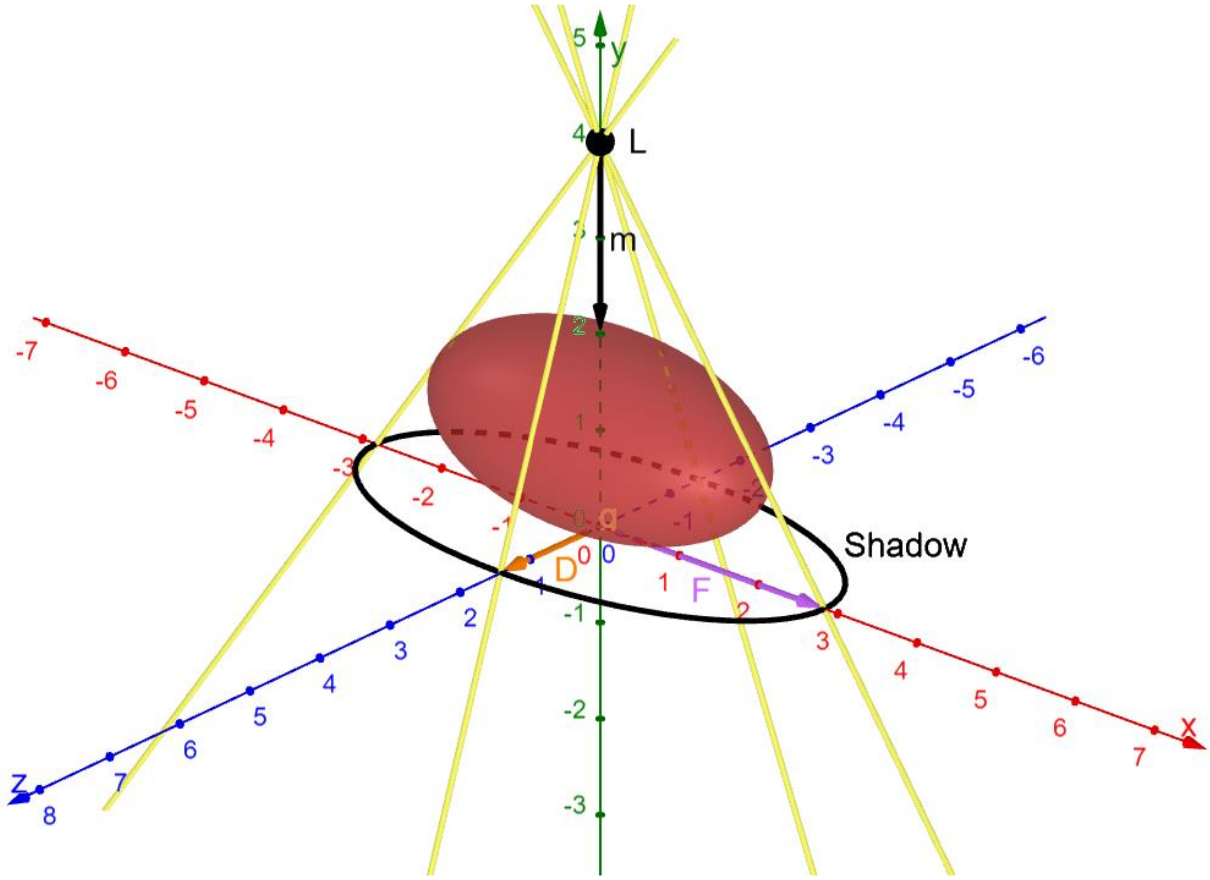

Figure 6.: Example of a elliptic shadow formed by the tangent lines (light rays) with its semi-minor and semi-major axes marked as D and F

The aim of the investigation is to find such function of *m* that describes the relationship between *m* and *D* and *F* because an area of a ellipse is defined as:

$$A_{ellipse} = \pi ab$$

where a and b are the lengths of the semi-major and semi-minor axes respectively (Toomer, 1990). In the case of the shadow the semi-major axis a will be *F* and semi-minor axis b will be *D*. Hence the function that we seek to derive is given by the equation:

$$f(m) = \pi \cdot F \cdot D$$

2. Plan for solving the problem

To determine the values of the semi-major axis F of the shadow and the semi-minor axis D we will need to consider two cases. The values of F and D are needed to calculate the area of the shadow.

- In the first case, we will consider the cross-section shown first in figure 4 with its corresponding tangent lines ("light rays"). Using trigonometry it is possible to find the semi minor axis D.
- In the second case, we will consider the cross-section first shown in figure 3 with its corresponding tangent lines ("light rays"). By using derivatives of the ellipse, the Fundamental Theorem of Calculus and the Similarity of Triangles Principle it is possible to generate a system of equations relating F to the values of x. To solve for m we find a relation of the derivative tangent function to the transformed equation of the ellipse. We then find zeros of such equation and substitute them into the previously given system of equations giving the relation of F to x.

3. Determining the function.

3.1 First case concerning a circle cross section.

In the first case we consider the situation in figure 7 (circle case) to determine the length of D.

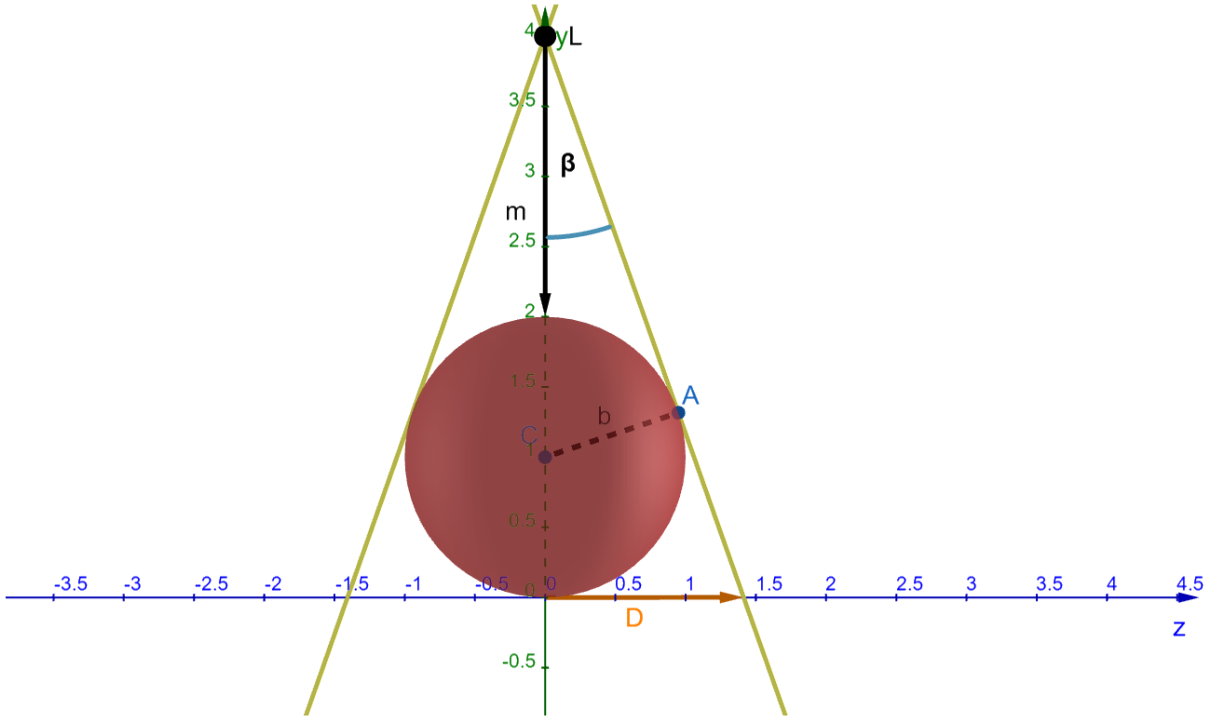

Figure 7.: Circle case needed to calculate D, where b=1. All other values shown are exemplary.

First we know from trigonometry that:

$$\sin(\beta) = \frac{1}{m+1}$$

This is because the tangent line creates a right angle with the radius of the circle and we know that the radius is equal to 1.

By rearranging:

$$m = \frac{1 - \sin(\beta)}{\sin(\beta)}$$

Where:

$$\beta \neq k\pi, k \in \mathbb{Z}$$

We exclude such values from the domain because such angles are impossible in this case and also would lead to dividing by 0.

Then again from trigonometry we know that:

$$\tan(\beta) = \frac{D}{2 + m}$$

We can rearrange this equation and substitute the first equation to it to obtain *D*.

$$\tan(\beta) \cdot \left(2 + \frac{1 - \sin(\beta)}{\sin(\beta)}\right) = D$$

This simplifies to:

$$D = \frac{\sin(\beta) + 1}{\cos(\beta)}$$

However now we must substitute the angle to find *D*. We can do this by taking the arcsine function of:

$$\sin(\beta) = \frac{1}{m + 1}$$

Which is equal to:

$$\arcsin\left(\frac{1}{m + 1}\right) = \beta$$

Now we substitute the arcsine function into to the previous equation. Now knowing that:

$$\sin(\arcsin(t)) = t$$

$$\cos(t) = \pm\sqrt{1 - \sin^2(t)}$$

We also know that the angle between *m* and the tangent line is always acute hence we only consider positive value of the square root. By using these previously stated identities we get:

$$D = \frac{\frac{1}{m + 1} + 1}{\sqrt{1 - \left(\frac{1}{m + 1}\right)^2}}$$

After simplification:

$$D = \frac{(2+m)|m+1|}{(m+1)\sqrt{(m+1)^2 - 1}}$$

This is the final solution in the circle case. This equation relates the length of the semi-minor axis of the shadow to the length of m which describes the height of the light.

3.2 Second case concerning a ellipse cross-section

In the second case we will consider the situation presented first in figure 3. To solve for F a system of equations with 4 equations and 5 unknowns describing the situation will be needed.

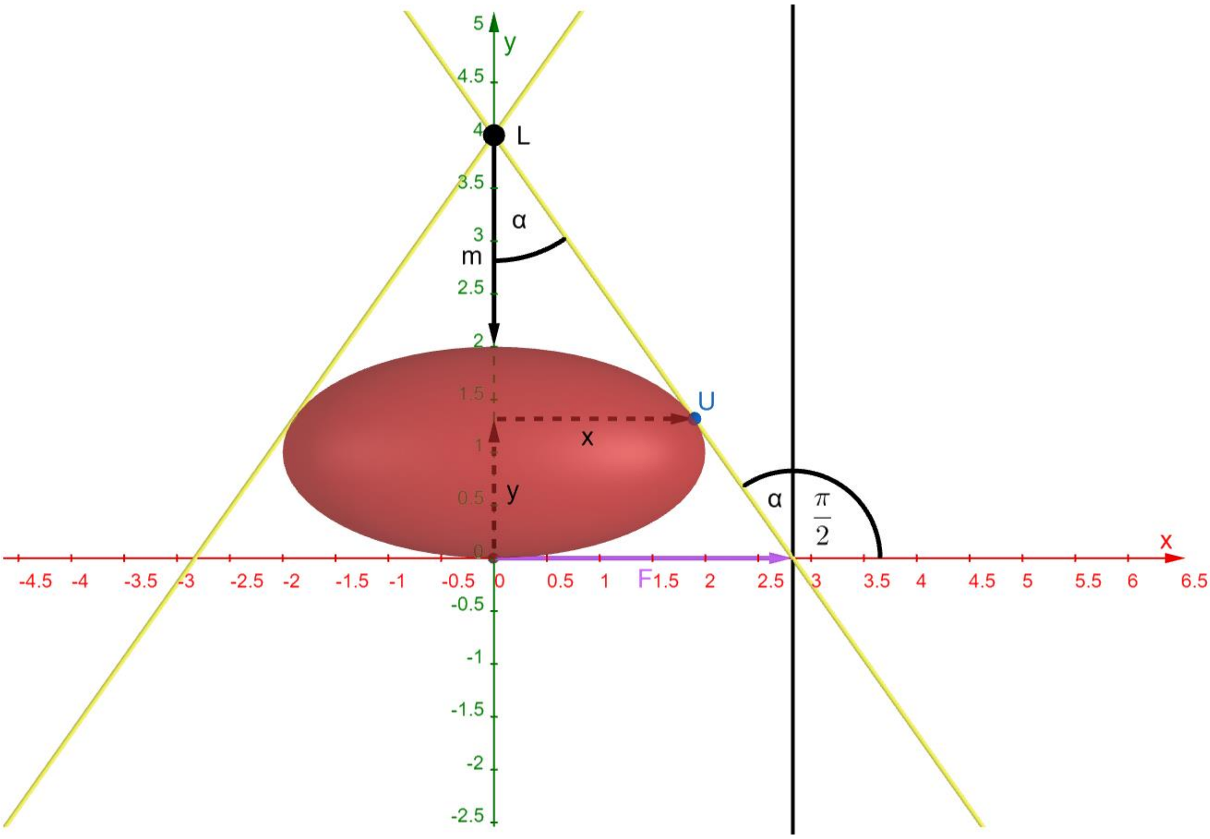

Figure 8.: Ellipse case, where all values shown on the axes are examlary.

Firstly we rearrange the equation of an ellipse:

$$\frac{x^2}{2^2} + \frac{(y-1)^2}{1^2} = 1$$

To yield:

$$y = \frac{\sqrt{4-x^2}+2}{2}$$

This equation relates the position of point U seen in figure 8 to the value of x. This equation will the first one in the system of equations. To obtain the second equation we can calculate the derivative of this function to yield a equation of the tangent line to the arbitrary point U.

$$\frac{d}{dx}\left(\frac{\sqrt{4-x^2}+2}{2}\right)$$

By taking out constants and applying the power rule in conjunction of the chain rule we get that:

$$\frac{\left(\frac{1}{2}(4-x)^{\frac{1}{2}-1}\cdot\frac{d}{dx}(4-x^2)\right)}{2}$$

This simplifies by again applying the power rule and taking out the constants to:

$$\frac{d}{dx}\left(\frac{\sqrt{4-x^2}+2}{2}\right) = \frac{-x}{2\sqrt{4-x^2}}$$

From the Fundamental Theorem of Calculus we know that the derivative is equal to the tangent line to a point on a curve. Using this and the fact the slope of a line is given by the tangent of the angle between the line and the x-axis (seen in figure 8) we determine that:

$$\tan(\alpha+\frac{\pi}{2}) = \frac{d}{dx}\left(\frac{\sqrt{4-x^2}+2}{2}\right) = \frac{-x}{2\sqrt{4-x^2}}$$

This yields a useful equation that will be used in the system of equations however:

$$\tan\left(\frac{\pi}{2}\right) = undefined$$

To simplify and improve clarity we change the tangent function using the co-function and even-odd trigonometric identities into cotangent function:

$$\tan(\alpha + \frac{\pi}{2}) = \cot(-\alpha)$$

$$\cot(-\alpha) = -\cot(\alpha)$$

Hence:

$$\tan(\alpha + \frac{\pi}{2}) = -\cot(\alpha)$$

This transformation is done to remove an angle sum from being used in the system of equations and to better present the value of the slope of the tangent line in relation to the angle it makes with the *y*-axis as shown in figure 8. Using the cotangent function allows to clearly see that the slope is negative.

Knowing this and the *y*-intercept of the tangent line of the curve we can derive the equation of a line that describes this situation. By substitution of *F* as a value of *x* seen in figure 8 we get a equation of a zero of that line:

$$0 = -\cot(\alpha) \cdot F + 2 + m$$

Now we must consider two triangles: *LNU* and *LWS* seen in figure 9. These two triangles are similar by the principle that all of them create the same angles on the inside.

We can use these triangles to obtain a ratio containing *F*. By using similarity of triangles principle of triangle *LNU* and *LWS* we obtain:

$$\frac{m + 2 - y}{2} = \frac{m + 2}{F}$$

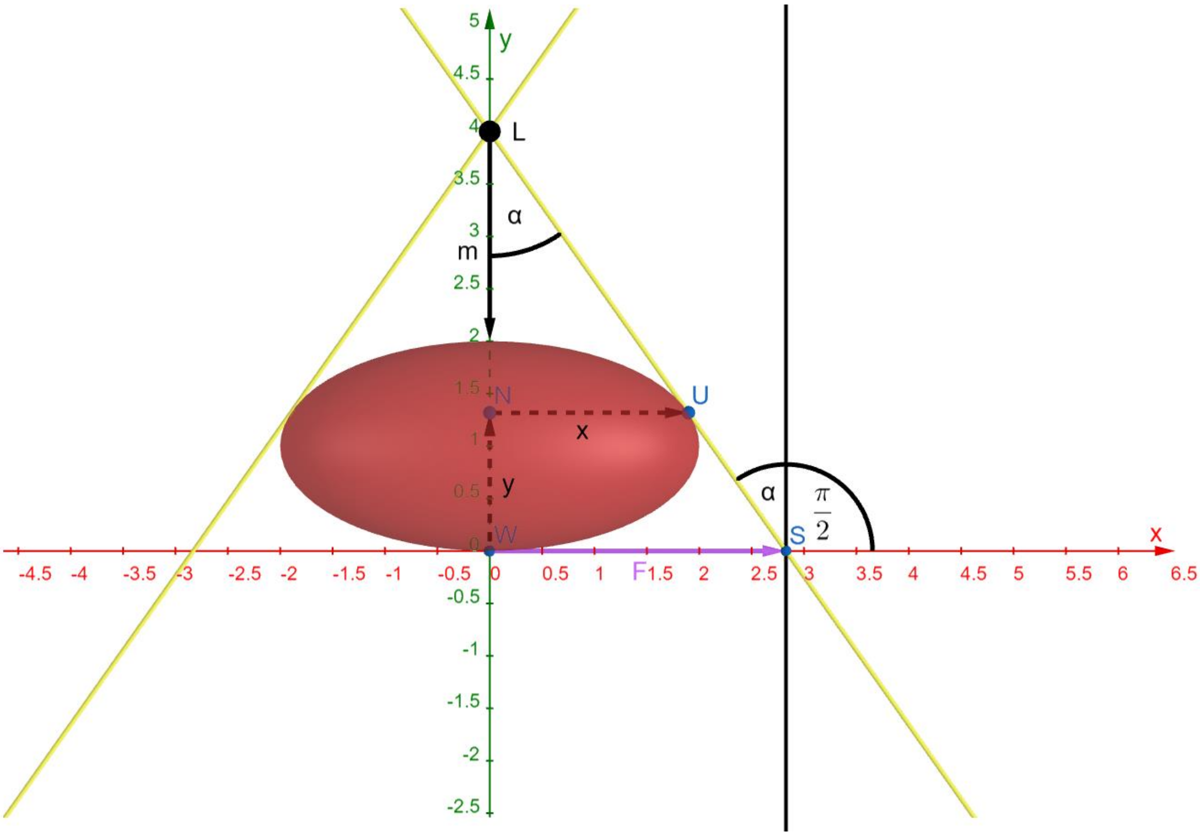

Figure 9.: Ellipse case with vertices of similar triangles

All these equations lead to a final system of equations which when solved will yield a function of m to F.

$$\begin{cases} \dfrac{m+2-y}{x} = \dfrac{m+2}{F} \\ \dfrac{x}{2\sqrt{4-x^2}} = -cot(\alpha) \\ 0 = -cot(\alpha)\,F + m + 2 \\ y = \dfrac{\sqrt{4-x^2}+2}{2} \end{cases}$$

Now we substitute the fourth equation into the first and the second to the third yielding:

$$\begin{cases} \dfrac{m+2-\dfrac{\sqrt{4-x^2}}{2}}{x} = \dfrac{m+2}{F} \\ 0 = \dfrac{x}{2\sqrt{4-x^2}}F + m + 2 \end{cases}$$

Now we can simplify further to obtain a equation of F in terms of x:

$$\begin{cases} \dfrac{2x(m+2)}{2m+4-\sqrt{4-x^2}} = F \\ -\dfrac{x}{2\sqrt{4-x^2}}F = m+2 \end{cases}$$

After substitution we get a equation relating F to x.

$$F = \frac{2x\left(-\frac{x}{2\sqrt{4-x^2}}F\right)}{2\left(-\frac{x}{2\sqrt{4-x^2}}F\right) - \sqrt{4-x^2}}$$

$$F = \frac{-\frac{x^2F}{\sqrt{4-x^2}}}{-\frac{xF+4-x^2}{\sqrt{4-x^2}}}$$

$$F = \frac{2x^2-4}{x}$$

To create a relation between F and the height of the light m we must use one more equation relating the line function with the slope given as a derivative equated to the equation of an ellipse that relates y to x.

$$\frac{\sqrt{4-x^2}+2}{2} = \frac{x^2}{2\sqrt{4-x^2}} + m + 2$$

After solving for m we obtain:

$$m = \frac{2-x^2-\sqrt{4-x^2}}{\sqrt{4-x^2}}$$

Now we need to expand this equation and get x in terms of m:

$$m\left(\sqrt{4-x^2}\right) + \sqrt{4-x^2} = 2-x^2$$

Now factorizing out the common term and squaring both sides to remove the square root we obtain:

$$\left(\sqrt{4-x^2}\right)^2 (m+1)^2 = (2-x^2)^2$$

$$x^4 - 4x^2 + 4 = 4m^2 + 8m + 4 - m^2x^2 - 2mx^2 - x^2$$

After further simplification we obtain a quadratic with a parameter:

$$-t^2 - t(m^2 + 2m - 3) + 4m^2 + 8m = 0$$

Where:

$$t = x^2$$

After solving this equation we should obtain 4 solutions of which only two positive should be considered as the two negative ones are not in the domain because we are working on distances. The discriminant of this quadratic is equal to:

$$\sqrt{\Delta_t} = \sqrt{m^4 + 4m^3 + 14m^2 + 20m + 9}$$

These operations yield us two zeros of the quadratic with a parameter which are:

$$t_1 = \frac{-m^2 - 2m + 3 + \sqrt{m^4 + 4m^3 + 14m^2 + 20m + 9}}{2}$$

$$t_2 = \frac{-m^2 - 2m + 3 - \sqrt{m^4 + 4m^3 + 14m^2 + 20m + 9}}{2}$$

Now using these zeros and imputing them into the original condition of:

$$t = x^2$$

We get 4 zeros of the original function:

$$x_1 = \frac{\sqrt{-2m^2 - 4m + 6 + 2\sqrt{m^4 + 4m^3 + 14m^2 + 20m + 9}}}{2}$$

$$x_2 = -\frac{\sqrt{-2m^2 - 4m + 6 + 2\sqrt{m^4 + 4m^3 + 14m^2 + 20m + 9}}}{2}$$

$$x_3 = \frac{\sqrt{-2m^2 - 4m + 6 - 2\sqrt{m^4 + 4m^3 + 14m^2 + 20m + 9}}}{2}$$

$$x_4 = -\frac{\sqrt{-2m^2 - 4m + 6 - 2\sqrt{m^4 + 4m^3 + 14m^2 + 20m + 9}}}{2}$$

The negative solutions are not in the domain hence they are disregarded in the next steps as well as x_3 which yields complex solutions. Now we substitute x_1 into:

$$F = \frac{2x^2 - 4}{x}$$

$$F = \frac{2\left(\frac{\sqrt{-2m^2 - 4m + 6 + 2\sqrt{m^4 + 4m^3 + 14m^2 + 20m + 9}}}{2}\right)^2 - 4}{\frac{\sqrt{-2m^2 - 4m + 6 + 2\sqrt{m^4 + 4m^3 + 14m^2 + 20m + 9}}}{2}}$$

Final determination of the function relating the area of the shadow to the value of *m*

Using the formula for the area of the ellipse:

$$A_{ellipse} = \pi ab$$

We can now obtain the function however we must take absolute value of both D and F as they are both distances in this case and cannot be negative. Substituting both D and F into the formula we obtain:

$$A_{ellipse} = |\pi DF|$$

$$A_{ellipse} = \left| \pi \cdot \frac{2\left(\frac{\sqrt{-2m^2 - 4m + 6 + 2\sqrt{m^4 + 4m^3 + 14m^2 + 20m + 9}}}{2}\right)^2 - 4}{\frac{\sqrt{-2m^2 - 4m + 6 + 2\sqrt{m^4 + 4m^3 + 14m^2 + 20m + 9}}}{2}} \cdot \frac{(2+m)|m+1|}{(m+1)\sqrt{(m+1)^2 - 1}} \right|$$

We can now graph the function using the GeoGebra online graphical calculator (https://www.geogebra.org/graphing?lang=pl):

Figure 10.: Graph of the function f(m) = $A_{ellipse}$

Analysis and interpretation of results

The function graphs the relationship between the height m and the area of the shadow created by the light rays. The logical geometric interpretation of the problem suggests that with m approaching infinity the area of the shadow would converge to the area of the original ellipse used to obtain the prolate spheroid. We can check if this is true by calculating the limit of this function to find if it converges on the value of the area of the original ellipse:

The area of the original ellipse:

$$A_{ellipse} = \pi \cdot 2 \cdot 1$$

$$A_{ellipse} \approx 6{,}28\ (3\ s.f.)$$

The degree of the rounding is given to 3 significant figures again as per convention.

$$\lim_{m\to+\infty}\left|\pi \cdot \frac{2\left(\frac{\sqrt{-2m^2-4m+6+2\sqrt{m^4+4m^3+14m^2+20m+9}}}{2}\right)^2-4}{\frac{\sqrt{-2m^2-4m+6+2\sqrt{m^4+4m^3+14m^2+20m+9}}}{2}} \cdot \frac{(2+m)|m+1|}{(m+1)\sqrt{(m+1)^2-1}}\right|$$

$$\lim_{m \to +\infty} \pi DF = 2\pi$$

$$\lim_{m \to +\infty} \pi DF \approx 6{,}28\ (3\ s.f.)$$

By performing these operations we prove that the limit of the function is equal to the expected value of the area. This fits the logical interpretation and the limit can be graphed as horizontal asymptote of this function:

Figure 11.: Graph of f(m) and it's limit shown as horizontal asymptote

We can clearly see the asymptotic behavior of the function as *m* approaches infinity. The function also exhibits asymptotic behavior when *m* is nearing zero. This can be explained by the area being infinitely large when *m* is zero because the tangent line would be parallel to the *x*-axis.

3. Summary

3.1 Conclusions

The aim of my investigation has been to determine the function of m that describes the relationship between m and the area of an elliptical shadow to model a theoretical representation of a real-life situation where a rugby ball casts a shadow when placed under a lamp. The exploration was successful in determining the function. The function is given by:

$$f(m) = \left| \pi \cdot \frac{2\left(\frac{\sqrt{-2m^2 - 4m + 6 + 2\sqrt{m^4 + 4m^3 + 14m^2 + 20m + 9}}}{2}\right)^2 - 4}{\frac{\sqrt{-2m^2 - 4m + 6 + 2\sqrt{m^4 + 4m^3 + 14m^2 + 20m + 9}}}{2}} \cdot \frac{(2+m)|m+1|}{(m+1)\sqrt{(m+1)^2 - 1}} \right|$$

The logical geometric interpretation of the problem fits the calculated values of the limit of the function. The behavior of the function meets expectations for such a problem and doesn't present any anomalies.

Determining the lengths of the semi-minor axis of shadow using the trigonometric approach proved to be easier than using Calculus. It would be unreasonably difficult to try and solve the circle case (figure 7) using Calculus as the trigonometric approach is way easier. However, in the case of the semi-major axis, it is very difficult and complicated to solve this elliptical case using only trigonometry. Using properties of curves, and their tangents by their interpretation in Calculus in conjunction with trigonometry is way easier to solve this case. This difficulty of solving using only trigonometry is due to the nature of the derivative of the equation describing an ellipse which is not linear compared with the linear derivative of an equation describing the circle.

3.2 Extensions

Possible extensions of this work could include manipulating the position of the light and changing the spheroid into an oblate spheroid which is a similar spheroid however it is created by rotation about the semi-minor axis of the ellipse. The function could also be used to determine the answers to questions like: For what m is the shadow twice bigger than the original ellipse? Another extension could be an attempt at the generalization of this problem for all values of a and b within a reasonable domain such that when substituted it creates an ellipse.

3.3 Evaluation

3.3.1 Strengths

A strength of the exploration is a lack of any approximations in terms of the projections of the light rays. All values used to obtain the function are exact and provide a clear template for how to solve such problems with differing values of a and b. Another definite strength is the solution to the ellipse case. It provides a template to create functions that plot zeros of derivatives interpreted as tangent lines to a curve dependent on the variable y-intercept of the tangent line.

3.3.2 Limitations and Improvements

One limitation concerning the exploration is the use of defined values of a and b as the semi-major and semi-minor axes. Due to this it would be difficult to generalize this function to all values of a and b within a reasonable domain. This could be addressed by following the same steps as the exploration did however without defining an exact numerical value of a and b and rather including a range of possible values of a and b for which an ellipse is created. Another limitation of this exploration is that it is rather theoretical and uses values for a and b which are unreasonable for real-world dimensions of a rugby ball making this exploration purely theoretical and with limited direct applications to the real-world. This could be simply addressed by taking accurate measurements of the dimensions of a rugby ball and using them in the calculations. One more limitation is the nature of the position of the spheroid. In the exploration, it is assumed that the longer semi-major axis is parallel to the x-axis. The function derived does not consider a situation where the semi-major axis would lie on the y-axis as this would yield a completely different shadow. This could be addressed by solving such a case consisting only of one elliptical case and yielding a function describing the circular shadow.

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