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Investigation of area and distances distortions on maps in cartography. Analysis of the Mercator projection model.

Introduction:

Cartography is the discipline dealing with the creation, distribution, and study of maps (ScienceDirect). The actual topographic surface of Earth is not an appropriate reference surface to perform cartographic analysis. Its shape is very complex due to its roughness - the presence of high mountains and deep oceans, which results in varying altitudes. A better reference Earth model is represented as a geoid. The geoid is a surface of constant potential energy that coincides with mean sea level over the oceans. It resembles a flattened sphere with numerous undulations caused by gravitational irregularities (Li & Götze, 2001). Nevertheless, the geoid shape is not precise enough for geodetic studies and the mathematically exact reference surface that fits the geoid shape is introduced. It is called the oblate ellipsoid of revolution - a shape obtained as a consequence of the ellipse rotation about its minor axis (Osborne, 2013).

The aim of this investigation is to explore the area and the distances distortion resulting from applying the Mercator projection as the model of Earth. The work will firstly introduce the idea of the representation of the latitude and the longitude on the sphere, as well as representing the distances between the points on the curved surface. Then, projecting a sphere into the rectangular plane will be explained. The analogical to the sphere model analysis of the cylindrical projection will be presented. The next part of the exploration will focus on providing the examples of the size and distances distortions.

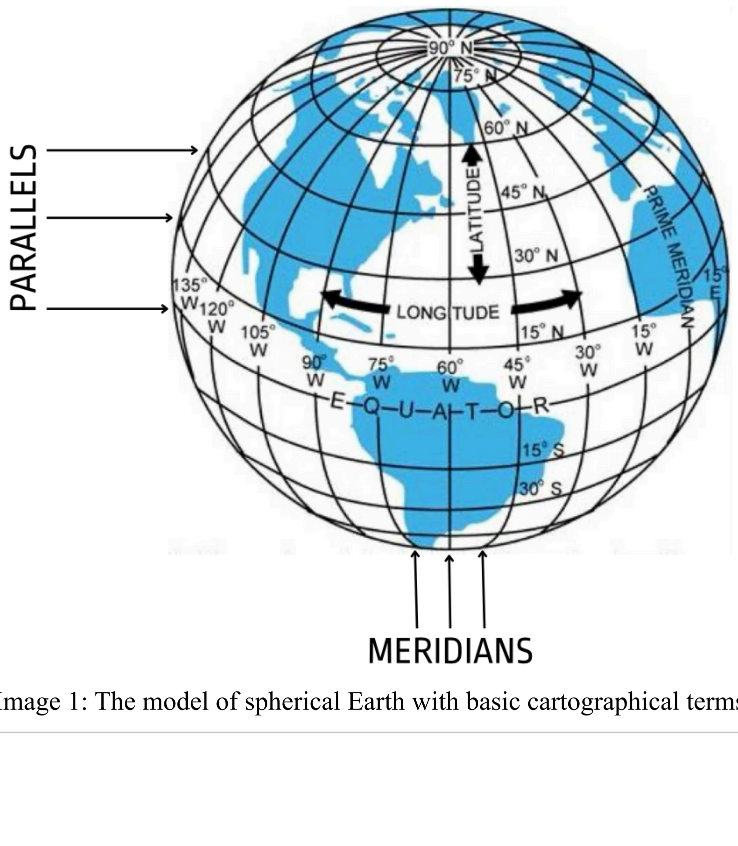

Image 1: The model of spherical Earth with basic cartographical terms

The basic cartographical terms used throughout the paper are: latitude ϕ, longitude λ, meridians and parallels. The following terms will be explained later, in the section regarding the Earth as a sphere. To remind the reader and enhance understanding, the image with Earth model with objects marked in the list above is presented (Image 1).

There are many ways to represent the latitude-longitude data indicated on three-dimensional shape by the two-dimensional map. A cartographic map is a projection defined by two functions $x(\phi,\lambda)$ and $y(\phi,\lambda)$, where (x, y) state the plane Cartesian coordinates corresponding to the latitude and longitude coordinates (ϕ, λ) (Osborne, 2013). Different projections are characterized by different properties - the most desirable features concern preservation of angles, area, shape and direction. For navigation systems the most important aspect is correct bearing - map projections that satisfy this condition are conformal map projections (Fletcher, 2023). Bearing is the angle in degrees measured clockwise from north. An example of the most commonly used conformal map projection is normal Mercator projection. It is used in aeronautical maps, nautical charts and maps representing time zones. It does not deform the real shapes and conserves true bearings. However, the disadvantage of this method is the distortion of sizes - as the objects approach the poles, they appear to be disproportionately larger compared to their real size and to the objects located closer to the Equator (Fletcher, 2023). The map below presents the true sizes (darker blue) of the countries compared to their appearance on the Mercator (lighter blue):

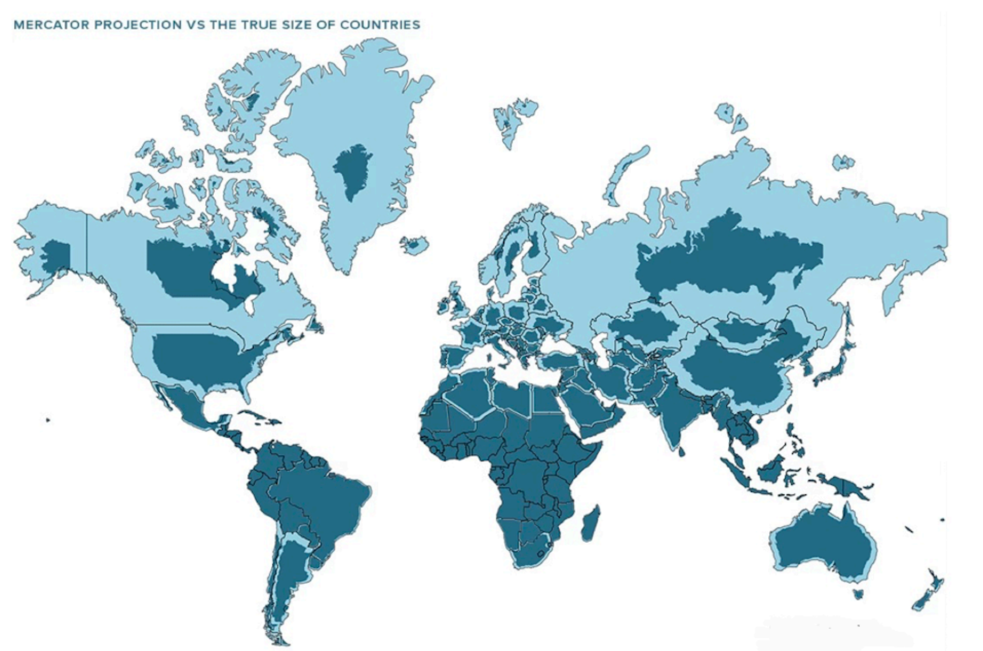

Image 2: The sizes of countries on the Mercator projection and their real size

The Earth as a sphere:

The Earth is an ellipsoid with a semi-major axis *a* (equatorial radius) and semi-minor axis *b* (Image 3).

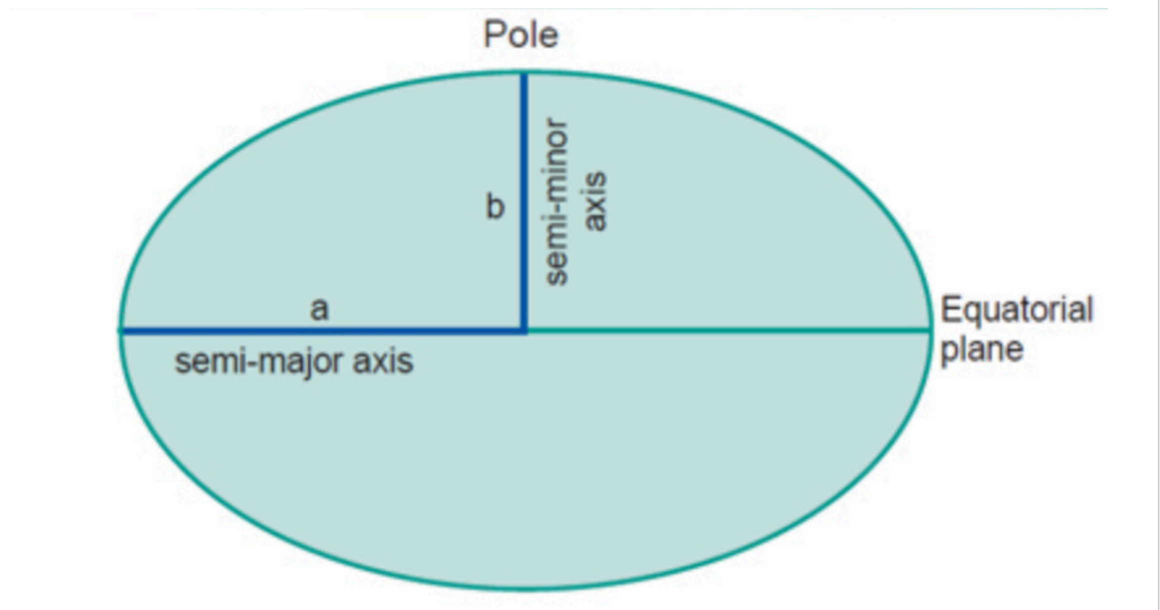

Image 3: The model of Earth as an ellipsoid

To approximate the radius *R* of the sphere obtained form an ellipsoid the triaxial arithmetic mean is taken (Osborne, 2013). Such approach (instead of geometric mean) is needed to reflect the fact, that Earth radius is greater when measured between two point on the Equator than when measuring between the northern and southern poles:

$$a = 6378.1370 \ km, b = 6356.7523 \ km$$

$$R = \frac{a+a+b}{3} = \frac{6378.1370 + 6378.1370 + 6356.7523}{3} \approx 6371 \ [km].$$

For this radius *R*, the circumference *C* equals:

$$C = 2\pi R = 2 \cdot \pi \cdot 6371 \approx 40030 \ [km]$$

Thus, one degree of latitude on the equatorial plane corresponds to:

$$\frac{40030}{360} = 111.2 \ [km].$$

For the model and calculations presented below, the assumption than an Earth is a unit sphere is introduced. Nevertheless for calculating the actual distances or lengths of some line segments, the scaling will be adjusted so that the radius of the sphere *r=1* matches the radius *R* of the Earth.

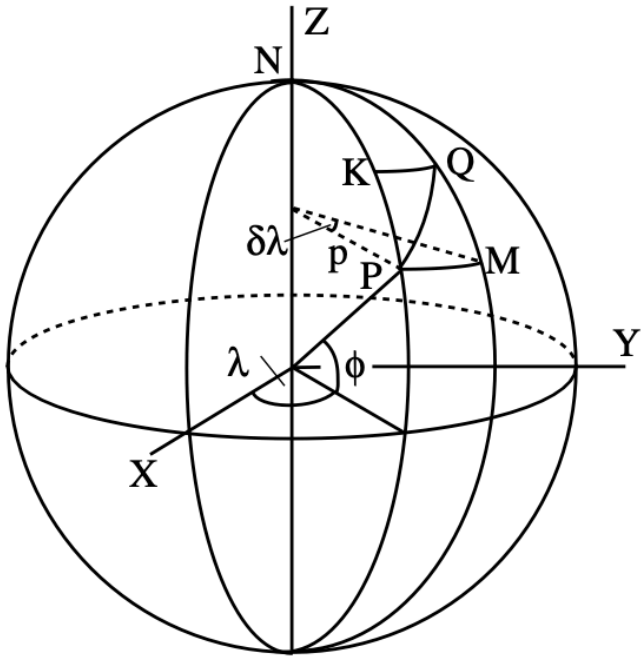

Image 4: The model of Earth as a sphere

The position of point P is considered (Image 4). Latitude ϕ is the angle between the point P and the point on the Equatorial plane, that has the same x coordinate as P. Its value belongs to the interval [−90°, 90°]. Longitude λ is the angle between the plane of the meridian passing through point P and the plane of the prime meridian. Its value belongs to the interval [-180°, 180°) (Osborne, 2013). The interval is opened, since the meridian 180° and -180° on a sphere would be exactly the same. Such approach prevents double counting the same longitude λ.

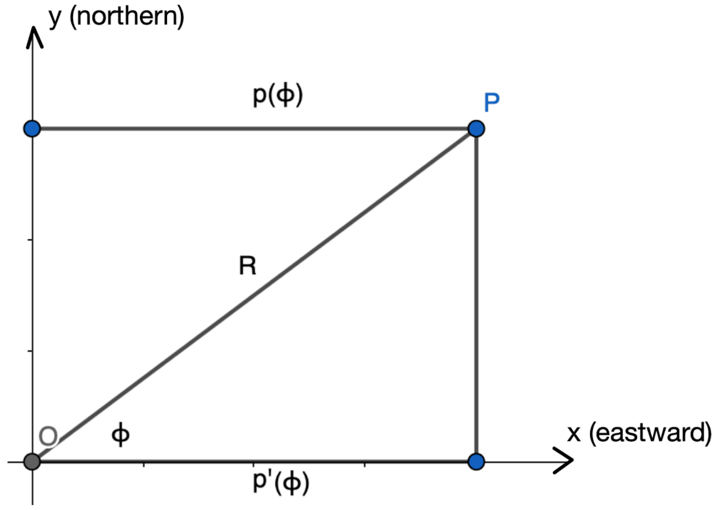

Image 5: Deriving the formula for the length of a small circle

The small circle is the circumference of the sphere parallel to the equatorial plane not being the equatorial radius. To calculate its radius, the latitude ϕ should be found by measuring the angle between the point *P,* which lies on the chosen small circle, and the point on the Equatorial plane, that has the same *x* coordinate as *P*. Its cosine value indicates the fraction by which the length of the Equator should be scaled. Since the model is a sphere, the radius *R* is sketch with an angle ϕ in relation to the Equator plane. From the diagram (Image 5), I derive the relation:

$$\frac{p(\phi)}{R} = cos\phi$$

$$p(\phi) = Rcos\phi \qquad [1]$$

where: *p(ϕ)* - the function of the radius of the small circle.

The sphere Earth can be placed in the three dimensional coordinate system.

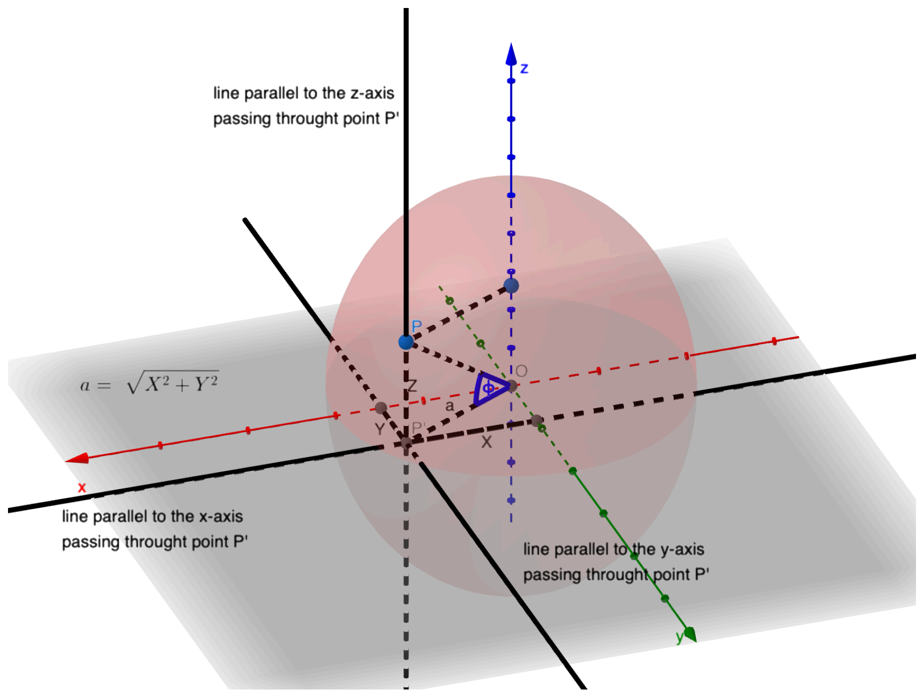

Image 6: Determining the latitude on the sphere

To find the latitude ϕ, the angle at the origin between the point *P* and the point *P'* on the *XY* plane - a point that has the same *x* and *y* coordinate as *P,* but *z* coordinate 0 - should be analyzed. To determine the latitude ϕ*,* its tangent will be calculated. It is the ratio between the opposite side *Z* - the vertical distance from the equatorial *XY* plane to the point *P* - and the distance *OP'.* To determine the length of *OP'*, the Pythagorean theorem should be used. The other sides of the triangle are X and Y - the distances from the origin O to the x and y coordinates of the point P' in the coordinate system.

From the diagram (Image 6), the relation is derived:

$$tan\phi = \frac{Z}{\sqrt{X^2 + Y^2}}$$

$$\phi = arctan(\frac{Z}{\sqrt{X^2 + Y^2}}) \qquad [2]$$

where ϕ - latitude,

X - the distance of the point P' from the origin to its x coordinate,

Y - the distance of the point P' from the origin to its y coordinate,

Z - the distance of the point P from the Equatorial plane to the north or to the south.

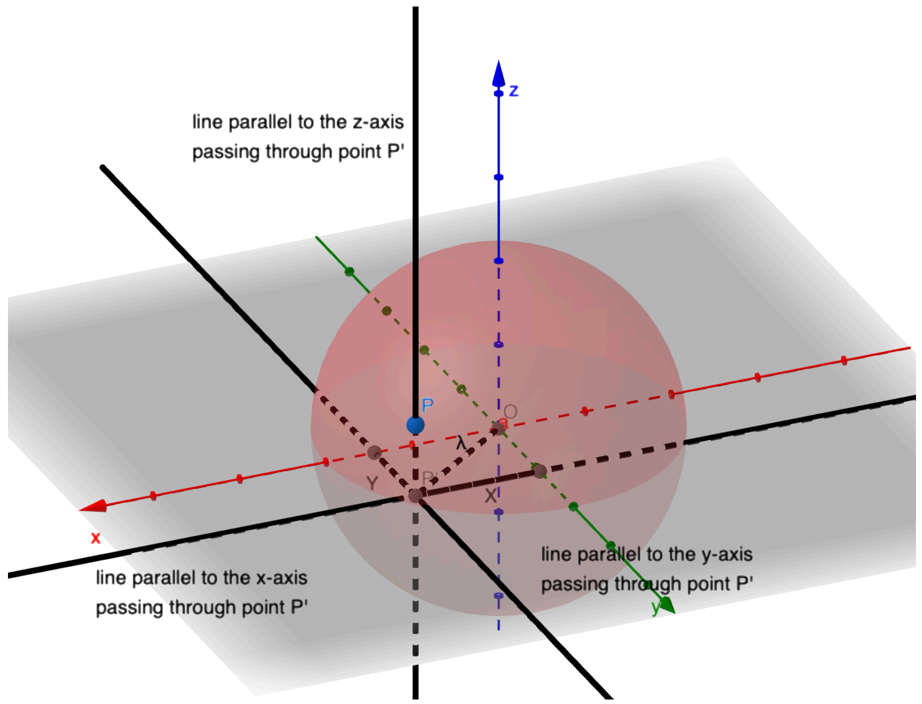

Image 7: Determining the longitude on the sphere

To find the longitude λ the angle between the prime meridian (0°) XZ plane and the plane of the meridian on which the point P lies should be calculated. The plane YZ represents the plane of the meridian 90°. Point P' preserves the same x and y coordinates as the point P, but it is transferred to the Equatorial plane, so its z coordinate equals 0. To find longitude λ, its tangent value will be determined. It is the ratio between the distance Y of the point P' from XZ plane to the east or to the west and its distance X from the YZ plane. From the diagram (Image 7), the relation is derived:

$$tan\lambda = \frac{Y}{X}$$

$$\lambda = arctan(\frac{Y}{X}) \qquad [3]$$

where λ - longitude.

Cylindrical projection:

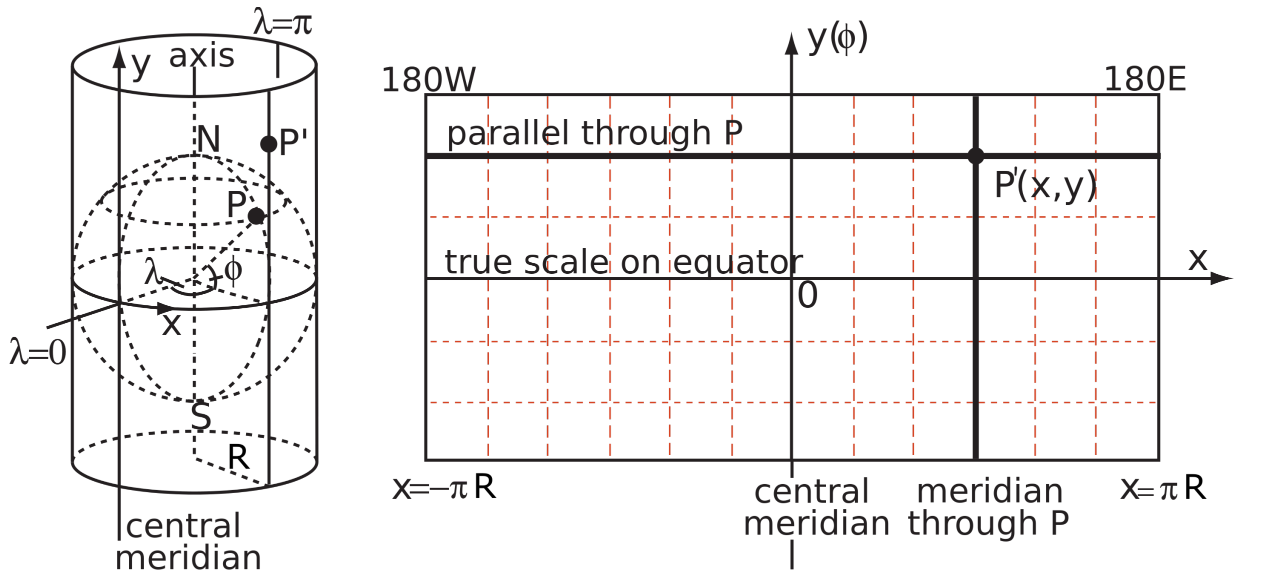

Image 8: The cylindrical projection

The idea of a cylindrical projection is to present a three-dimensional sphere as a two-dimensional rectangle. The sphere is inscribed into the cylinder (represented on the Image 8), whose radius is the same as the equatorial radius *R*. Only the lateral area of a cylinder is considered. The generator of the cylinder is the meridian at which the model is „cut" and unfolded. It is stated at λ=*180°*. After that, the cylinder forms a two-dimensional, rectangular map. The *x-axis* on the projection corresponds to the circumference of the Equator on the globe. The *y-axis* on the projection represents a prime meridian and its length corresponds to the Earth the diameter between the northern and southern poles. To obtain a projection, the meridians are projected on the plane according to the formula (Miller, 1942):

$$X = R\lambda\frac{\pi}{180}$$

where: X - the spacing of meridians on the projection.

The projection enforces the spacing of parallels described as (Miller, 1942):

$$Y = R \cdot ln(tan(45^\circ + \frac{\phi}{2})) \qquad [4]$$

where: Y - the spacing of parallels on the projection.

Since the transformations listed above are not the aim of the investigation and will not be directly used in calculations they will not be analyzed in details. Nevertheless, the future problem stated in this paper will derive conclusions from that.

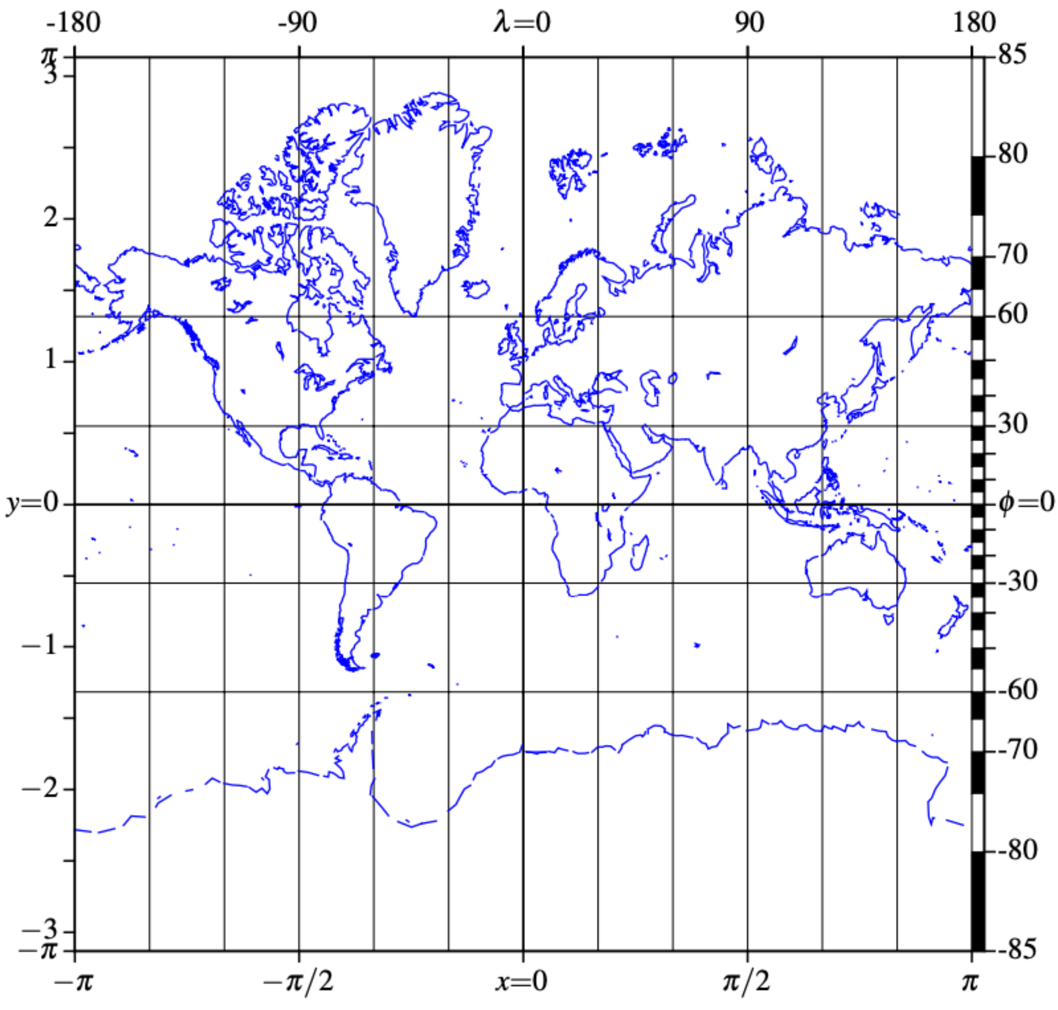

Image 9: The Mercator projection map

When it comes to Cartesian coordinates, the fundamental origin for Mercator is the point O on the equator, *x-axis* includes the eastward direction of the projected equator and the positive *y-axis* is taken as the northern area of projected meridian through O. Any point on the Earth model can be denoted as an ordered pair of coordinates (ϕ , λ) (Osborne, 2013). Mercator wanted his projection to satisfy the following conditions: the bearings are preserved, the north-south direction is vertical, the east-west direction is horizontal and the length of the equator is preserved (Vezie, 2016).

On the sphere, all meridians intersect at the poles, but on normal cylindrical projection they do not intersect at all. Hence, the projection extents to infinity in north-south y direction, which can be checked using the Equation 4:

$$Y = 6371 \cdot ln(tan(45^\circ + \frac{90^\circ}{2})) = 6371 \cdot ln(tan(45^\circ + 45^\circ)) = 6371 \cdot ln(tan(90^\circ))$$

$$tan 90^\circ = undefined$$

$$f(90^\circ) \rightarrow +\infty$$

$$Y = 6371 \cdot ln(tan(45^\circ + \frac{-90^\circ}{2})) = 6371 \cdot ln(tan(45^\circ - 45^\circ)) = 6371 \cdot ln(tan(0^\circ))$$

$$ln 0 = undefined$$

$$f(-90^\circ) \rightarrow -\infty$$

The implication of that property is the truncation of the Mercator projection at the latitude smaller than 90°/-90°. The areas in the regions approaching poles become uncontrollably enlarged and deformed - which is visible when looking at Antarctica (Image 9).

On the projection, all parallels are of equal width, so all of them are of the same length as the equator. However, on the sphere the true length of parallel corresponds to $R cos\phi$ (Equation 1). The parallel scale factor k is the ratio of the length of a parallel on the projection to the corresponding true length. It is equal to:

$$k = \frac{R cos 0^\circ}{R cos\phi} = \frac{R \cdot 1}{R cos\phi} = \frac{1}{cos\phi} = sec\phi, \quad [5]$$

where: k - the parallel scale factor.

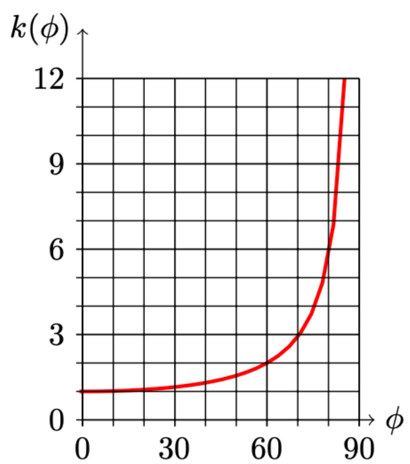

Image 10: The function of the parallel scale factor k

The parallel scale factor k measures how distances along the parallels (east-west direction) are distorted compared to their true size on Earth. It increases as the latitude ϕ increases (Image 10). The only parallel, where there is no distortion is the equator. There, the latitude ϕ is 0°, so:

$$k = sec\,0^{\circ} = 1$$

It means that the length on the projection exactly corresponds to the length on the sphere. The parallel scale factor k increases to infinity as the poles are approached, since their latitude ϕ is undefined (Image 10). The practical implication of that is using the projections other than Mercator for the polar maps.

As the parallels on the Earth approaches the poles, the distances between them decreases (Image 1). However, on the Mercator their spacing differs (Image 9). Thus, despite the fact, that the meridians preserve their length, the distances measured along them in relation to parallels are distorted. To express y coordinate Y on the Mercator projection, the function including latitude ϕ is calculated. It implies the vertical scale factor h. The projection is conformal, so it preserves the angle between two curves that cross each other on Earth on the projection. One of the assumptions of the conformal map is that its parallel scale factor k is the same as the vertical scale factor h (Rajput, 2024). Thus, using the Equation 5:

$$h = k = sec\phi \qquad [6]$$

Image 11: Greenland and DRC areas on the Mercator projection

The real world example that can be used to represent the distortions implied by the scale factors *k, h* can be analyzed comparing the area of Greenland (2 166 000 km^2) and Democratic Republic of Congo (2 345 000 km^2). In fact, their true size is very similar. The Democratic Republic of Congo is bigger than Greenland by:

$$\%\,\Delta_{size} = \frac{2\ 345\ 000 - 2\ 166\ 000}{2\ 345\ 000} \cdot 100\,\% \approx 7.63\,\%$$

The Image 11 represents the Mercator projection. The area labelled „Grenlandia" shows the dimensions of Greenland on the Mercator projection, while the greenish shape is its real size. The blue shape is the real size of the Democratic Republic of Congo.

To calculate the area that is displayed on the Mercator projection, the area scale factor $\mu_A(\phi)$ is used. It informs by how much the true area was enlarged after displaying in the projection. It is expressed as the product of parallel *k* and vertical *h* scale factors (Equation 6):

$$\mu_A(\phi) = k \cdot h = sec(\phi) \cdot sec(\phi) = sec^2(\phi) \quad [7]$$

where $\mu_A(\phi)$ - area scale factor.

$$A_{Mercator} = \mu_A(\phi) \cdot A_{True} \quad [8]$$

Where: $A_{Mercator}$ - area displayed on the Mercator projection,

A_{True} - the true area of an object.

To find the latitudes ϕ needed for the calculations, the center of each region will be determined. To find it, the middle of the latitude ϕ between the most northern and most southern point of the region will be found. I will find the coordinated of this points using an online website (Source 2).

For Greenland:

$$\frac{83.63^\circ + 59.77^\circ}{2} = 71.70^\circ$$

For Democratic Republic of Congo:

$$\frac{5.38^\circ + (-13.46)^\circ}{2} = -4.04^\circ$$

The center of Greenland is placed approximately at the latitude ϕ of 71.70°. The scale factor of area is (Equation 7):

$$\mu_A(\phi) = sec^2(71.70^\circ) \approx 10.143$$

Democratic Republic of Congo area crosses the equator and is placed much further from the poles. The center of the continent is placed around latitude ϕ of 10°. Thus, its scale factor of area equals (Equation 7):

$$\mu_A(\phi) = sec^2(-4.04^\circ) \approx 1.005$$

Thus the areas displayed on the Mercator projection are (Equation 8):

Greenland:

$$A_{Greenland\ on\ the\ Mercator} = 10.143 \cdot 2\ 166\ 000 \approx 21\ 969\ 738\ [km^2]$$

Democratic Republic of Congo:

$$A_{DRC\ on\ the\ Mercator} = 1.005 \cdot 2\ 345\ 000 \approx 2\ 356\ 725\ [km^2]$$

The following results implies that the area of Greenland on the Mercator projection is over 10 times exaggerated compared to the area of Democratic Republic of Congo and the actual area on the sphere. Democratic Republic of Congo area on the Mercator projection does not vary much from the actual size.

<u>Distances on the Mercator projection:</u>

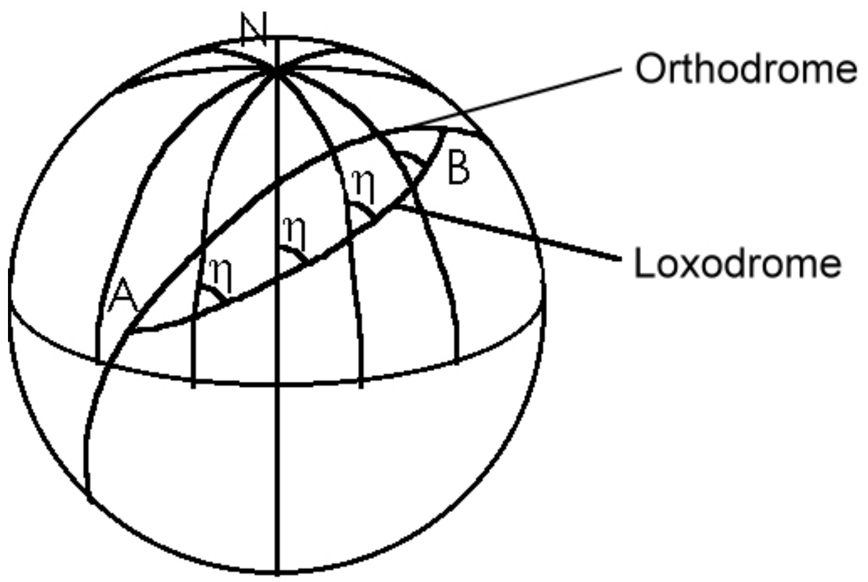

Image 12: The loxodrome and orthodrome on the sphere

The distance can be calculated either along a loxodrome or along an orthodrome. A loxodrome is a curve that cuts all meridians of a rotating surface at the same angle (Kos et al., 1999). Thus, by traveling along the loxodrome, the correct bearing is preserved. An orthodrome is the intersection of the sphere with a plane containing the centre of the sphere (Petrović, 2013). When determining the distances on the Mercator projection, the path of a loxodrome is used, while when determining the distanced on the sphere, the path of an orthodrome is used. The distances determined using orthodrome are always shorter or equal to loxodrome distances. For example, the route between

Paris and New York by the loxodrome track takes approximately 5295 km, while by the orthodrome 5069 km (Bertici & Herbei, 2014).

1. Calculating the distance on the Mercator projection (along the loxodrome)

To calculate the distance $|AB|$ on the Mercator projection, the length of the arc of the loxodrome should be found. Nevertheless, the derivation of the formula requires implying differential geometry, which is beyond the topic investigated. The formula for the distance between the points *ST* is (Kos et al., 1999):

$$|ST|_M = R \cdot \frac{[(90^\circ - \phi_2) - (90^\circ - \phi_1)] \cdot \frac{\pi}{180^\circ}}{cos\mu} \quad [9]$$

where: $|ST|_M$ - the distance between points *S* and *T* on the Mercator projection

μ - angle at which the loxodrome cuts the meridians.

The $cos\mu$ value in the expression determines by how much is the distance on the Mercator projection elongated compared to the true distance.

2. Calculating the distance on the sphere (along the orthodrome)

To calculate the distance $|ST|$ along the orthodrome, the spherical rule of cosines is used. The theorem states, that the sides and angles of spherical triangles are analogous to the ordinary cosine rule from plane trigonometry. It is given by the equation (Roy, 2022):

$$\alpha = arccos(sin\phi_1 sin\phi_2 + cos\phi_1 cos\phi_2 cos(\Delta\lambda)) \quad [10]$$

$$|ST|_S = \frac{\alpha}{360^\circ} \cdot C, \quad [11]$$

where: $|ST|_S$ - the distance between points *S* and *T* on the sphere,

α - the central angle.

The spherical rule of cosine formula is explained below:

The Image 13 represents the two points on the sphere *S, T* with marked latitudes ϕ_1, ϕ_2 and the difference between their longitudes $\Delta\lambda$. The points *S, T* can be parametrized, so that their 3D Cartesian coordinates *X, Y, Z* are represented using latitude and longitude relation (Osborne, 2013):

$$S(cos\phi_1 cos\lambda_1,\ cos\phi_1 sin\lambda_1,\ sin\phi_1)$$

$$T(cos\phi_2 cos\lambda_2,\ cos\phi_2 sin\lambda_2,\ sin\phi_2)$$

Let *S* and *T* be treated as a vectors:

$$s = \begin{pmatrix} cos\phi_1 cos\lambda_1 \\ cos\phi_1 sin\lambda_1 \\ sin\phi_1 \end{pmatrix},$$

$$t = \begin{pmatrix} cos\phi_2 cos\lambda_2 \\ cos\phi_2 sin\lambda_2 \\ sin\phi_2 \end{pmatrix}.$$

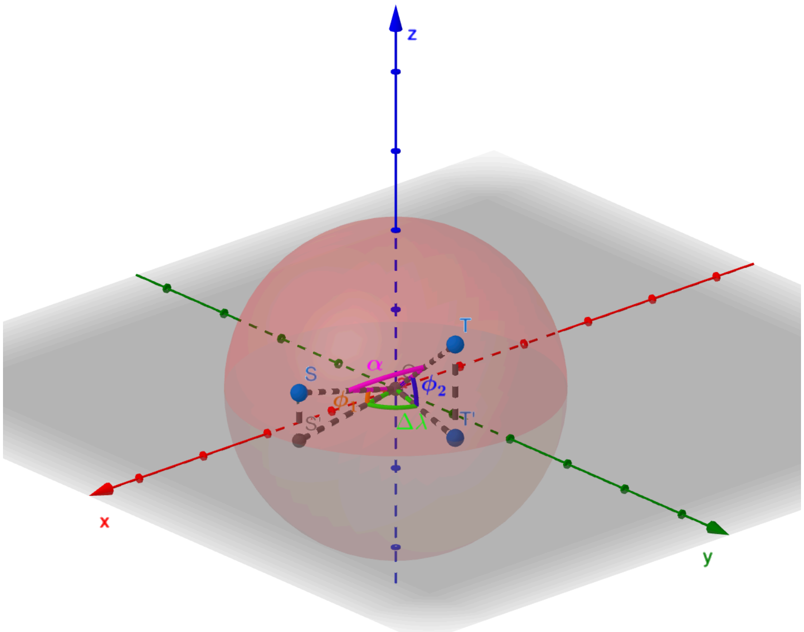

Image 13: The scenario for calculating the distance on the sphere

The angle α, between two vectors can be found using:

$$cos\alpha = \frac{\boldsymbol{s} \cdot \boldsymbol{t}}{|\boldsymbol{s}||\boldsymbol{t}|}$$

Using the assumption, that the unit sphere is analyzed, the lengths of the vectors ***s, t*** are equal to 1. Thus, the formula simplifies to:

$$cos\alpha = \frac{\boldsymbol{s} \cdot \boldsymbol{t}}{1 \cdot 1} = \boldsymbol{s} \cdot \boldsymbol{t}$$

After substitution, the formula presents as follows:

$$cos\alpha = \begin{pmatrix} cos\phi_1 cos\lambda_1 \\ cos\phi_1 sin\lambda_1 \\ sin\phi_1 \end{pmatrix} \cdot \begin{pmatrix} cos\phi_2 cos\lambda_2 \\ cos\phi_2 sin\lambda_2 \\ sin\phi_2 \end{pmatrix} = (cos\phi_1 cos\lambda_1)(cos\phi_2 cos\lambda_2) + (cos\phi_1 sin\lambda_1)(cos\phi_2 sin\lambda_2) + (sin\phi_1)(sin\phi_2)$$

$$cos\alpha = cos\phi_1 cos\lambda_1 cos\phi_2 cos\lambda_2 + cos\phi_1 sin\lambda_1 cos\phi_2 sin\lambda_2 + sin\phi_1 sin\phi_2 = cos\phi_1 cos\phi_2 (cos\lambda_1 cos\lambda_2 + sin\lambda_1 sin\lambda_2) + sin\phi_1 \sin\phi_2$$

Using trigonometric identity:

$$cos(\lambda_2 - \lambda_1) = cos\lambda_1 cos\lambda_2 + sin\lambda_1 sin\lambda_2,$$

The formula simplifies to:

$$cos\alpha = sin\phi_1 sin\phi_2 + cos\phi_1 cos\phi_2 cos(\Delta\lambda)$$

$$\alpha = arccos(sin\phi_1 sin\phi_2 + cos\phi_1 cos\phi_2 cos(\Delta\lambda)) \qquad [10]$$

The central angle α is used as an inscribed angle to find an arc of the great circle that indicates the distance on the orthodrome. To leave the assumption that the Earth is a unit sphere, the length of an arc is multiplied by the circumference of the actual Earth dimension *C*.

The real world example that will show the distance distortions between two places is presented below:

The approximate coordinates of the cities are:

San Francisco, United States - S(37.8°, -122.4°)

Moscow, Russia - M(55.8°, 37.6°)

For clearer visualization, the placement of cities on the sphere is represented by the diagram below:

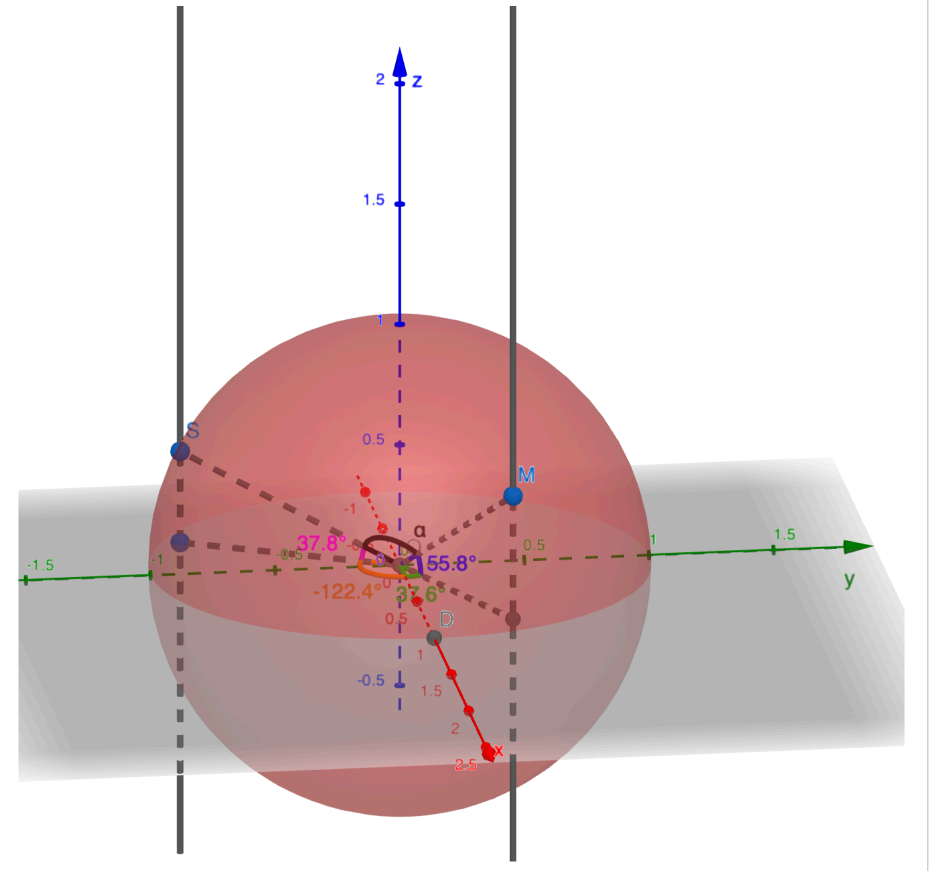

Image 14: Determining the distance between Moscow and San Francisco

The difference between the latitudes $\Delta\phi$ and longitudes $\Delta\lambda$ between those two cities are:

$$\Delta\lambda = |\lambda_1 - \lambda_2| = 37.6^\circ - (-122.4^\circ) = 160.0^\circ,$$

$$\Delta\phi = |\phi_1 - \phi_2| = 55.8^\circ - 37.8^\circ = 18.0^\circ.$$

To calculate the distance along the orthodrome, the spherical rule of cosine is used (Equation 10):

$$\alpha = arccos(sin 55.8^\circ sin 37.8^\circ + cos 55.8^\circ cos 37.8^\circ cos 160.0^\circ = 84.86^\circ$$

Thus, the distance along the orthodrome is (Equation 11):

$$|SM|_s = C \cdot \frac{\alpha}{360^\circ} = 40030 \cdot \frac{84.86^\circ}{360^\circ} \approx 9436\ [km]$$

To calculate the distance |*SM*| along the loxodrome, the angle μ at which the loxodrome cuts the meridians is needed. Since the preservation of angles on the Mercator projection is not the topic of the work, the angle μ was found using an online website (Source 1) and measures 78.87°.

Thus, the distance on the projection is (Equation 9):

$$|ST|_M = R \cdot \frac{[(90^\circ - \ 37.8^\circ) - (90^\circ - 55.8^\circ)] \cdot \frac{\pi}{180^\circ}}{cos 78.87^\circ} \approx 10370\ [km]$$

The difference between the distances Δd calculated is:

$$\Delta d = |ST|_M - |ST|_S = 10370 - 9436 = 934\ [km].$$

The distance on the Mercator projection (along the loxodrome marked with red) is longer than the distance on the sphere (along the loxodrome marked with yellow), which is illustrated on the image below (Image 15):

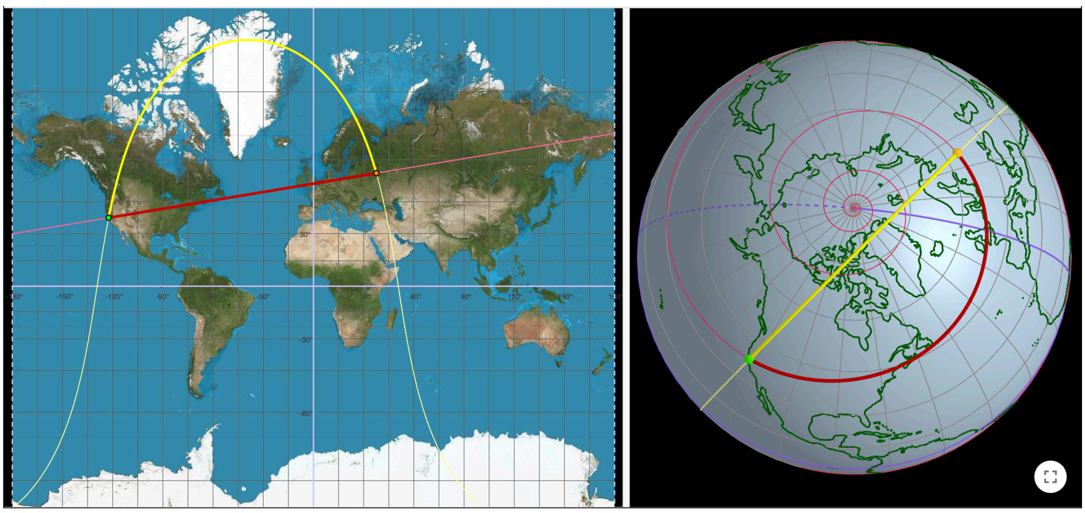

Image 15: The comparison of the distance between Moscow and San Francisco dependently on the route (loxodrome and orthodrome)

The correctness of the calculations can be supported by the external sources. A publication by Bertici & Herbei (Bertici & Herbei, 2014) suggests that the orthodromic distance should be 9476 km and the loxodromic distance should be 10051 km. The percentage error of the calculations above is:

For the distance along the orthodrome:

$$percentage\ error = \frac{9476 - 9436}{9476} \cdot 100\,\% = 0.422\,\%$$

For the distance along the loxodrome;

$$percentage\ error = \frac{|10051 - 10370|}{10051} \cdot 100\,\% = 3.17\,\%$$

The most probable sources of errors is discussed in the discussion section.

Area sizes on the Mercator projection:

The approach of calculating the area on the sphere depends on the size and the shape of the object. Since the analysis of the geometry of a sphere is not the aim of the work, the piece sufficient for the further comparison and analysis would be a relatively small, rectangular object. The formula for its area will be derived step by step. Such shape of the object was chosen, since the approach is clear for representation and will apply also in squared figures.

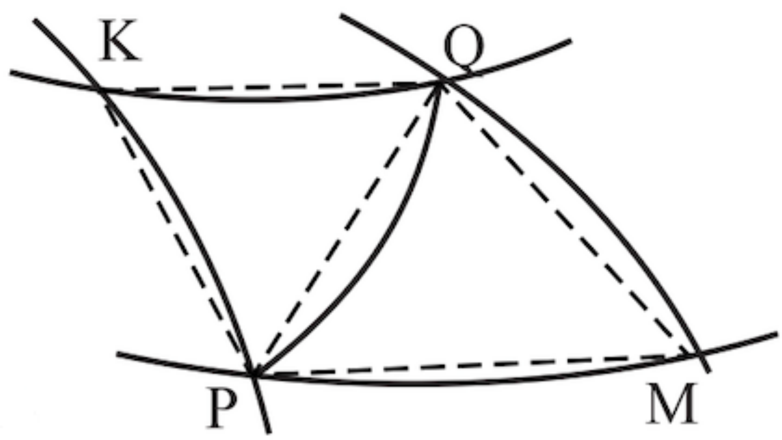

Image 16: The features of the area KQMP on the sphere

For the small areas, the deviation of the curved area from the corresponding area on the plane is negligible. The areas between the arcs, marked with solid lines, and the cords, marked with dashed lines, between any two points chosen are infinitesimal - they are so small, that they can be ignored throughout calculations (Image 16). Such areas can be mapped without using any projections (Osborne, 2013).

The Cartesian coordinates of the points *K, Q, P, M* in the three dimensional system are:

$$K(X_K, Y_K, Z_K)$$

$$Q(X_Q, Y_Q, Z_Q)$$

$$P(X_P, Y_P, Z_P)$$

$$M(X_M, Y_M, Z_M)$$

The following image represents an object *KQPM* on the sphere (Image 17).

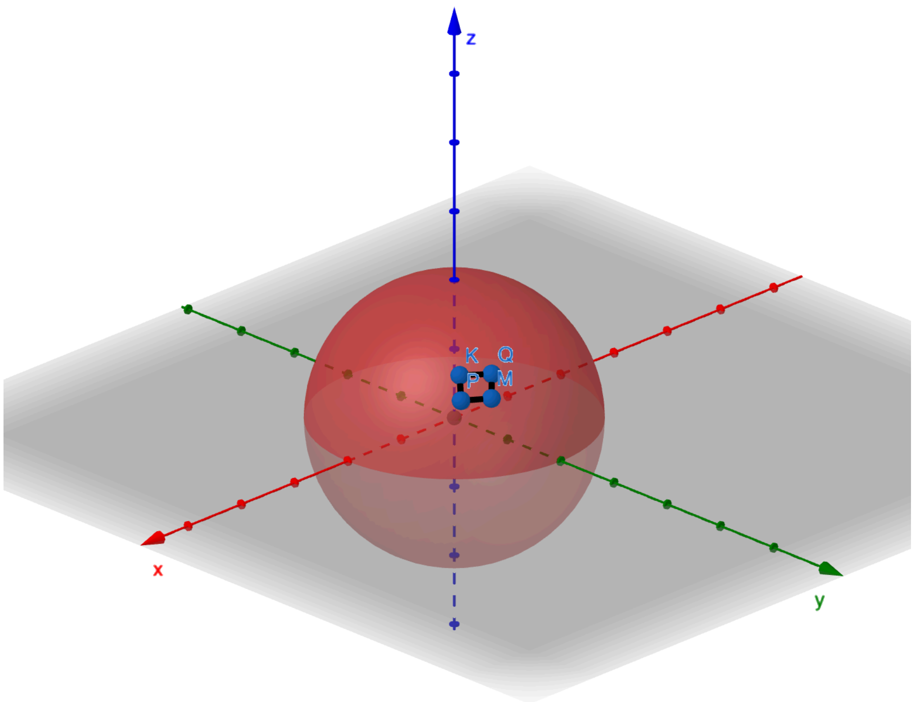

Image 17: The area KQMP on the sphere model

The latitude ϕ and longitude λ presented as a set of coordinates (ϕ , λ) for each of the points is (Equation 2 & 3):

$$K(\phi_K, \lambda_K) = K(arctan(\frac{Z_K}{\sqrt{X_K^2 + Y_K^2}}), arctan(\frac{Y_K}{X_K}))$$

$$Q(\phi_Q, \lambda_Q) = Q(arctan(\frac{Z_Q}{\sqrt{X_Q^2 + Y_Q^2}}), arctan(\frac{Y_Q}{X_Q}))$$

$$P(\phi_P, \lambda_P) = P(arctan(\frac{Z_P}{\sqrt{X_P^2 + Y_P^2}}), arctan(\frac{Y_P}{X_P}))$$

$$M(\phi_M, \lambda_M) = M(arctan(\frac{Z_M}{\sqrt{X_M^2 + Y_M^2}}), arctan(\frac{Y_M}{X_M}))$$

Nevertheless, to improve readability the notation (ϕ_X , λ_X) will be kept.

Since the object is a rectangle, the pair of sides *KQ* and *PM* and the second pair *KP* and *MQ* will be parallel. Thus:

$$\phi_K = \phi_Q$$

$$\phi_P = \phi_M$$

$$\lambda_K = \lambda_P$$

$$\lambda_Q = \lambda_M$$

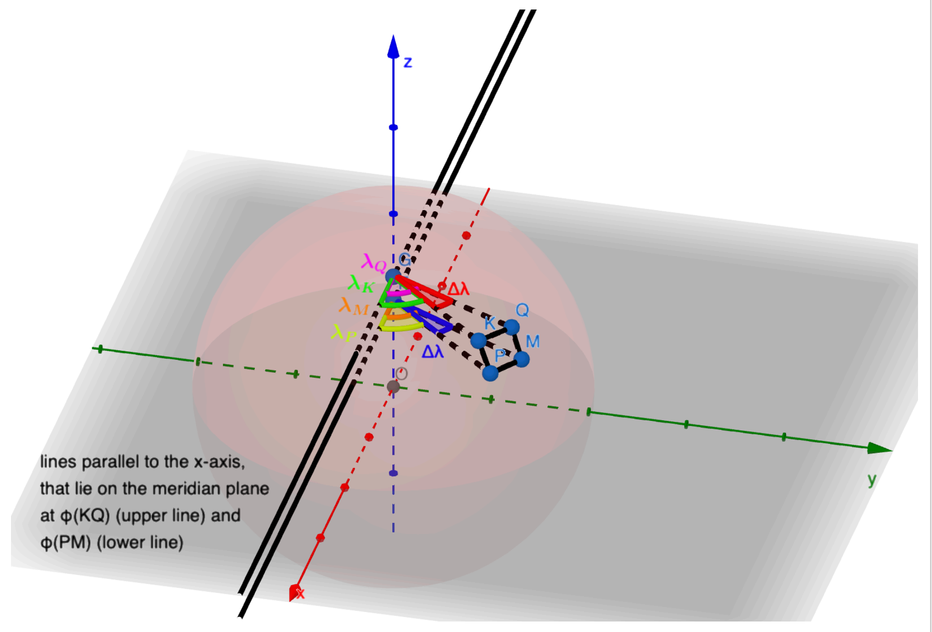

Image 18: The length of KQ/QM on the sphere model

To calculate the area of *KQPM,* the lengths of the sides need to be calculated. Since*,* the object is a rectangle the difference between the longitudes $\Delta\lambda$ is the same (Image 18). It can be determined using the formula:

$$\Delta\lambda = \Delta\lambda_{KQ} = \Delta\lambda_{PM} = |\lambda_Q - \lambda_K| = |\lambda_M - \lambda_P|.$$

Despite the fact, that *KQ* and *PM* lie on different parallels, the difference of their lengths will be negligible. To calculate their lengths, the length of the small circle *p(ϕ)* that they lie on should be calculated (Equation 1), and then the fraction of the parallel that the segment occupies must be determined. This relation is presented using the formula:

$$|KQ| = |PM| = 2\pi p(\phi_{KQ})\frac{\Delta\lambda_{KQ}}{360^\circ} = 2\pi R cos\phi_{KQ}\frac{\Delta\lambda_{KQ}}{360^\circ}$$

where: |*KQ*| - length of a line segment.

To calculate the area of the rectangle, also the length of the second side *KP* or *QM* should be found (Image 19).

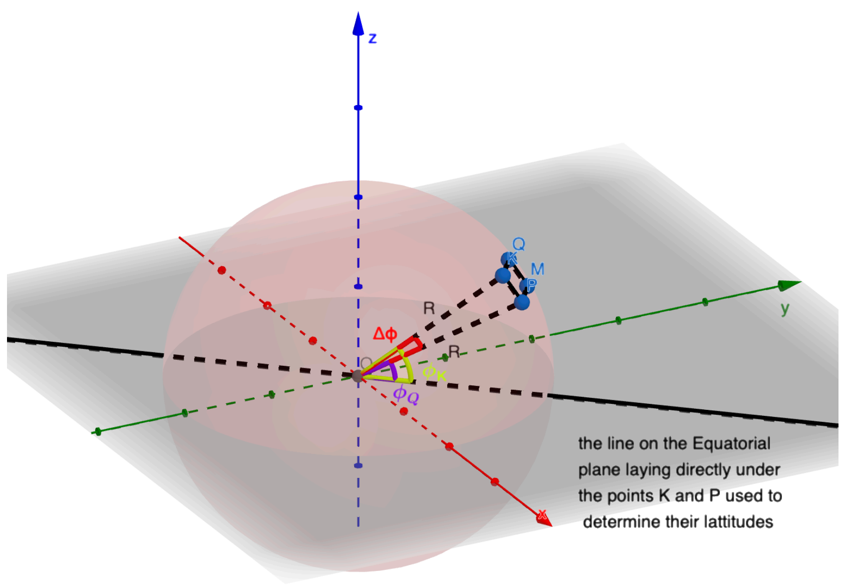

Image 19: The length of KP/QM on the sphere model

Since the model is a sphere, the distance form the origin *O* to the points *K* and *P* will be the same and its length will equal the length of a radius *R*. Hence, the triangle *KOP* will be isosceles. The value of the angle *KOP* is the difference between the latitudes *ϕ* of the points *K* and *P*:

$$\angle KOP = \Delta\phi = \phi_{KQ} - \phi_{PM}$$

Thus, the value of the angles *OKP* and *OPK* is:

$$\angle OKP = \angle OPK = \frac{180^\circ - \angle KOP}{2}$$

Hence, using the sine rule, the length of a segment *KP* is:

$$\frac{\angle OKP}{R} = \frac{\angle KOP}{|KP|}$$

$$|KP| = \frac{R \cdot \angle KOP}{\angle OKP}$$

From the previous analysis, the general formula for the area A of a small rectangular object of the sphere is derived:

$$A = |KP| \cdot |KQ| = (\frac{\Delta\phi \cdot R}{\frac{180^\circ - \Delta\phi}{2}}) \cdot (2\pi R cos\phi_{K/Q/M/P} \cdot \frac{\Delta\lambda}{360^\circ}) \quad [12]$$

where: A - area of an object.

To check the correctness of the formula, the area of a rectangular Galeria Krakowska shopping mall in Cracow will be analyzed. The image below represents the object of interest on the map (Image 20):

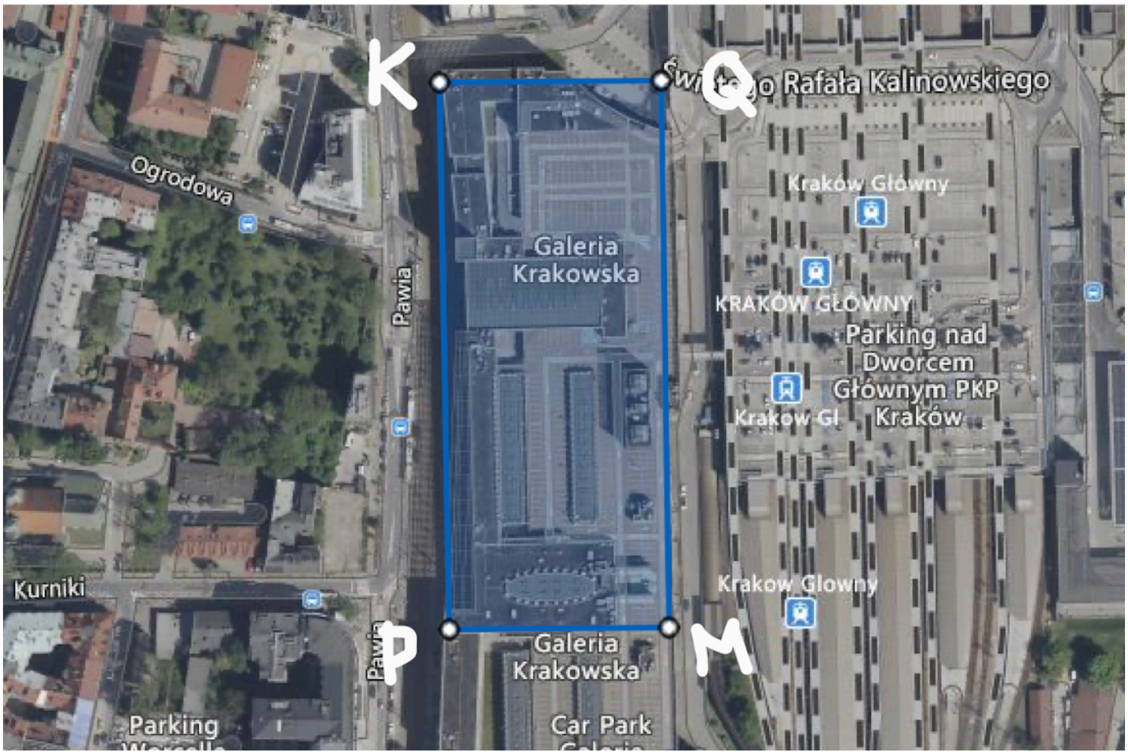

Image 20: Galeria Krakowska on the map

Using CalcMaps website (Source 2), coordinates of the edges of the mall are found:

$K(50.0686^\circ,\ 19.9452^\circ)$

$Q(50.0686^\circ,\ 19.9467^\circ)$

$P(50.0660^\circ,\ 19.9452^\circ)$

$M(50.0660^\circ,\ 19.9467^\circ)$

The differences between the latitudes $\Delta\phi$ and longitudes $\Delta\lambda$ of the points are:

$$\Delta\phi = 50.0686° - 50.0660° = 0.0026°$$

$$\Delta\lambda = 19.9467° - 19.9452° = 0.0015°$$

So its area A is (Equation 12):

$$A = (\frac{0.0031° \cdot 6371}{\frac{180° - 0.0031°}{2}}) \cdot (2\pi \cdot 6371 \cdot cos(50.0691°) \cdot \frac{0.0015°}{360°}) \approx 19\ 706\ [m^2]$$

The data for the area of three levels of commercial venues provided by online sources is approximately 60 000 m^2 (Source 3). So for the one level, the area A_{level} is:

$$A_{level} \approx \frac{60\ 000}{3} \approx 20\ 000\ [m^2]$$

The percentage error of the calculations is:

$$percentage\ error \approx \frac{20000 - 19706}{20000} \cdot 100\,\% \approx 1.47\,\%$$

The most probable sources of errors will be discussed in the discussion section.

The land sizes on the Mercator projection are distorted in relation to their true sizes (those measured on the sphere), which is a consequence of equal spacing of parallels on the projection (Equation 3). Let the same object *KQPM* be considered. The line segments *KQ* and *PM* are sketched along the parallel. Thus, after projection their true length will be elongated according to the parallel scale factor k:

$$|K'Q'| = |KQ| \cdot k = |KQ| \cdot sec(\phi_1)$$

$$|P'M'| = |PM| \cdot k = |PM| \cdot sec(\phi_2)$$

The influence of the vertical scale factor h on the line segment *PK* is checked:

$$|K'P'| = |KP| \cdot h = |KP| \cdot sec(\phi_2 - \phi_1)$$

Thus, the area of the *K'Q'P'M'* measured on the projection is:

$$A_{K'Q'P'M'} = |K'P'| \cdot |K'Q'|$$

The area on the projection is larger than the area measured on the sphere. The percentage by which the projection was enlarged can be calculated:

$$\%\,\Delta A = \frac{A_{K'Q'P'M'} - A_{KQPM}}{A_{KQPM}} \cdot 100\,\%$$

Discussion and conclusions:

When it comes to the distortion of the areas, the Mercator projection maintains accurate sizes near the equator, which can be supported by analysis of the area of Democratic Republic of Congo. As the distance from the equator increases, the distortion also increases, which is a consequence of increasing parallel scale factor k (Image 10). Thus, the Mercator is not an accurate projection to determine the true size of an object. The size of an object is highly dependent on its latitude ϕ. It might also cause confusions when it comes to comparing the areas, as presented on the Image 11. Apart from analyzing the objects on the Mercator projection, the work has explored the aspect of negligibility of the variations between the areas of the small objects on the sphere and their projections (Image 16). However, the approach explored in this work does not apply to greater areas.

When it comes to the distortions of the distances, on the Mercator projection they are determined along the loxodrome, while on the sphere the distance is determined along the orthodrome. The distances on the loxodrome are equal to the distances on the orthodrome only in two cases. The first one is along the meridians - the angle at which the loxodrome crosses the meridians μ then is 0. Thus, the formula for the distance simplifies as the value of cosine is 1 (Equation 9). It results in no elongation of the distance. The second one is the distance along the equator. All parallels on the Mercator projection are of equal length of the Earth circumference C, so the length of a pathway of a distance on the sphere and on the projection is exactly the same. When it comes to determining the distances located along the parallel different than the Equator or between the points located on different latitudes and longitudes, the distance on the loxodrome is longer. The greater the distance between two points, the greater the absolute difference between the distance measured along the loxodrome and along the orthodrome.

The percentage errors obtained in the calculations related with real world examples are low, which indicates the accuracy of the results. That also implies that the formulae invented work. The fluctuations in the length of the distances might be caused by the precision of the Earth radius R used or by choosing the coordinates of the cities. In this exploration the one in the city centers were picked, however in the publication used for the comparison there was no information about the actual coordinates chosen. Moreover, the distance along the loxodrome might be impacted by the precision of the bearing found. The distortion of the area size of a shopping mall can be caused by an inaccuracy in determining the coordinates of the edges. Also the webpage of reference does not provide such information.

The main attribute of this work is the fact, that most of the formulae were either derived or proofed. It is worth mention, that the topic explored is of high practical relevance. Maps are the tools that are used daily - understanding the fundaments of the Mercator projection and the distortions associated with it expands the awareness of the representation of the world and helps with its evaluation. However, the limitation of this work is not operating on the differential geometry. If it was applied, in case of calculating the areas on the sphere, the results yield might be more accurate. It will also allow for enhancing the understanding and proving the formula for the distance on the loxodrome.

The possible extension of the research might be exploring other properties of the conformal map projection such as preservation of bearings after the projection or comparing the analyzed parameters of the Mercator projection with other type of projection, for example Lambert's equal area projection.

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Images:

Image 1: *https://www.dauntless-soft.com/products/freebies/library/books/ak/8-2.htm* [Access date: 2 Dec 2024]

Image 2: *https://www.visualcapitalist.com/mercator-map-true-size-of-countries/* [Access date: 2 Dec 2024]

Image 3: *http://kartoweb.itc.nl/geometrics/* [Access date: 2 Dec 2024]

Image 4: Osborne (2013)

Image 5: Generated using: *https://www.geogebra.org/calculator* [Access date: 2 Dec 2024]

Image 6: Generated using: *https://www.geogebra.org/calculator* [Access date: 2 Dec 2024]

Image 7: Generated using: *https://www.geogebra.org/calculator* [Access date: 2 Dec 2024]

Image 8: Osborne (2013)

Image 9: Osborne (2013)

Image 10: Osborne (2013)

Image 11: Generated using: *https://www.thetruesize.com/* [Access date: 2 Dec 2024]

Image 12: *https://commons.wikimedia.org/wiki/File:Loxodrome2.jpg* [Access date: 2 Dec 2024]

Image 13: Generated using: *https://www.geogebra.org/calculator* [Access date: 2 Dec 2024]

Image 14: Generated using: *https://www.geogebra.org/calculator* [Access date: 2 Dec 2024]

Image 15: Generated using: *https://www.geogebra.org/m/rkwzupqc* [Access date: 2 Dec 2024]

Image 16: Osborne (2013)

Image 17: Generated using: *https://www.geogebra.org/calculator* [Access date: 2 Dec 2024]

Image 18: Generated using: *https://www.geogebra.org/calculator* [Access date: 2 Dec 2024]

Image 19: Generated using: *https://www.geogebra.org/calculator* [Access date: 2 Dec 2024]

Image 20: Generated using: *https://www.calcmaps.com/* [Access date: 2 Dec 2024]

Online Sources:

Source 1: Used for calculating the bearing: https://www.dcode.fr/azimuth [Access date: 2 Dec 2024]

Source 2: Used for determining the coordinates of the points: https://www.calcmaps.com/ [Access date: 2 Dec 2024]

Source 3: The reference area of Galeria Krakowska https://www.urbanity.pl/malopolskie/krakow/galeria-krakowska,b714 [Access date: 2 Dec 2024]